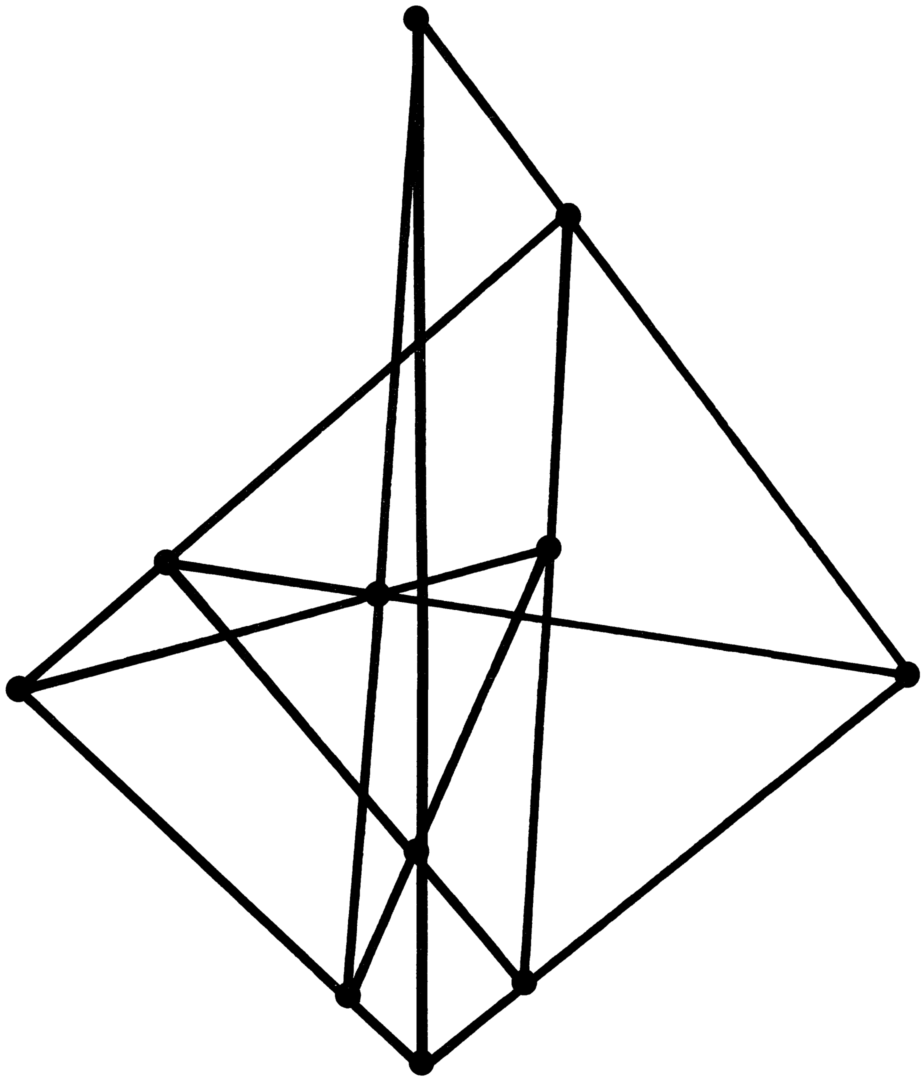
MATHEMATICS MAGAZINE



Vol. 57 No. 2
March 1984

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The Mathematical Association of America
1529 Eighteenth Street, N.W.
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MATHEMATICS MAGAZINE

THE MATHEMATICAL ASSOCIATION OF AMERICA

Volume 57, Number 2, March 1984
(ISSN— 0025-570X)

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AUTHORS

Michael Doob ("Applications of Graph Theory in Linear Algebra") was introduced to graph theory by the late Moses Richardson and to its algebraic aspects by Alan Hoffman at the Graduate Center of the City University of New York. After receiving his Ph.D. from that institution in 1969, he joined the Mathematics Department at the University of Manitoba. He is a co-author with Dragos Cvetković and Horst Sachs of the book *Spectra of Graphs,* and, having published about thirty papers in the area, has a continuing research interest in the algebraic aspects of graph theory.

ILLUSTRATIONS

The illustration for "Sum of Squares," p. 92, is reproduced from "Pyramid, pile, and sum of squares," *Historia Mathematica,* 8 (1981) 64, with permission from Academic Press.

Vic Norton illustrated the riddle, p. 94.

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The MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Montpelier, VT, five times a year: January, March, May, September, and November.
The annual subscription price for the MATHEMATICS MAGAZINE to an individual member of the Association is $11 included as part of the annual dues. (Annual dues for regular members, exclusive of annual subscription prices for MAA journals, are $22. Student, unemployed and emeritus members receive a 50% discount; new members receive a 30% dues discount for the first two years of membership.) The non-member/library subscription price is $28 per year. Bulk subscriptions (5 or more copies) are available to colleges and universities for classroom distribution to undergraduate students at a 41% discount ($6.50 per copy—minimum order $32.50).
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ARTICLES

Applications of Graph Theory in Linear Algebra

Graph-theoretic methods can be used to prove theorems in linear algebra.

MICHAEL DOOB
The University of Manitoba
Winnipeg, Manitoba, Canada R3T 2N2

Graph theory has existed for many years not only as an area of mathematical study but also as an intuitive and illustrative tool. The use of graphs in wiring diagrams is a straightforward representation of the physical elements of an electrical circuit; a street map is also a graph with the streets as edges, intersections of streets as vertices, and street names as labels of the edges. The graphs resemble the physical object that they represent in these cases, and so the application (and sometimes the genesis) of the graph-theoretic ideas is immediate. A flow diagram of a computer program and a road map with one way streets are examples of graphs which contain the concept of direction or flow to the edges; these are called directed graphs.

There are applications of graphs and directed graphs in almost all areas of the physical sciences and mathematics, many of them known for fifty years or more, but very few of these ideas have percolated down to the undergraduate student. The purpose of this article is to apply graph-theoretic ideas to some of the fundamental topics in linear algebra. While there are many such applications, we shall focus on only two of the most elementary ones, i.e., matrix multiplication and the theory of determinants. The presentation is usable as a supplement to the usual classroom lectures; in fact, this paper grew out of notes used in a linear algebra course for sophomores.

The main tool to be used is the directed graph. Intuitively this can be thought of as a set of points (or vertices) with arrows (or arcs) joining some of the points. A label may be put on an arc. More formally, a **digraph** consists of a set of vertices V and a subset of ordered pairs of vertices called the **arcs**. A **labelling** of the digraph is a function from the arcs to the real numbers. A labelled digraph is usually visualized by considering the vertices as points with arcs as arrows going from vertex i to vertex j whenever (i, j) belongs to the sets of arcs. The ith vertex of the arc (i, j) is called its **initial vertex**, while the jth vertex is called its **terminal vertex**. The arc is then given a label which is the image of that arc under the labelling function. When the initial and terminal vertices are identical, the arc is called a **loop**. We shall sometimes say that an arc "goes out of" its initial vertex and that it "goes into" its terminal vertex. The number of arcs that go out of a vertex is called its **outdegree**, and the number of arcs that go into that vertex is called its **indegree**.

A digraph G is a **subgraph** of a digraph H if the vertices and arcs of G are contained in the set of vertices and set of arcs of H. One type of subgraph of H is a **walk**: this consists of a sequence of vertices $v_0, v_1, v_2,\dots,v_n$ such that (v_{i-1}, v_i) is an arc in H for $i=1,2,\dots,n$. We use $(v_0, v_1, v_2,\dots,v_n)$ to denote such a walk (which is consistent with the notation for an arc); we then say that the **length** of the walk is n. A **path** is a walk where all the vertices are distinct, and a **cycle** has $v_1, v_2,\dots,v_n$ distinct and $v_0 = v_n$. Another type of subgraph is a **factor**; it has the indegree and outdegree of each vertex equal to one. A moment's thought reveals that a factor is simply a (vertex) disjoint union of cycles. The **weight** of a subgraph, $W(G)$, is the product of the labels of the arcs in that subgraph. Weights of walks, paths, cycles, and factors are similarly defined.

Matrix multiplication and the König digraph

In this section we examine a particular directed graph associated with an $m \times n$ matrix A. The multiplication of two matrices is equivalent to "glueing" their digraphs together to form a single graph. An entry in the product matrix is then related to the weights of certain paths in the new graph. Most standard proofs about matrix multiplication involve the manipulation of subscripts and/or the interchanging of summations. These techniques, while valid, tend to obscure the underlying ideas; this is avoided by the graph-theoretic approach.

Let $A = (a_{ij})$ be a matrix with m rows and n columns. The **König digraph**, $G(A)$, is a labelled digraph with $m + n$ vertices—m of these vertices correspond to the rows and n correspond to the columns. The arc from vertex i to vertex j has a_{ij} as its label. When drawing the graph we shall omit arcs with a zero label; this will clarify the relationships between several matrix and graphic operations. As a further convention, we shall draw the row vertices on the left and the column vertices on the right. The top vertex will correspond to the first row or column, the one below it to the second, etc. An illustration of a matrix A and its König digraph $G(A)$ is shown in FIGURE 1. D. König [5] used this digraph in his well-known and fundamental book in 1936; he referred to it tangentially even earlier [4] in 1916.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \end{pmatrix}$$

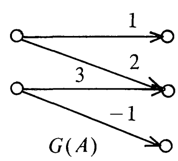

FIGURE 1. **A 2×3 matrix A and its König digraph $G(A)$.**

If A and B are both $m \times n$ matrices, then $G(A)$ and $G(B)$ are essentially the same digraph with different labels. The matrix sum $A + B$ is defined, and the digraph $G(A + B)$ obviously is obtained by adding the weights on the corresponding arcs of $G(A)$ and $G(B)$. The usual additive properties of matrices carry over directly to the digraphs. The proofs of these familiar properties, e.g., the associative, commutative, and distributive laws, are identical to the usual ones, but rather than focusing on a particular row and column entry in a matrix, a label of a particular arc is used to determine the validity of these properties. The multiplication of a matrix by a scalar is also easy to visualize. The digraph $G(rA)$ is obtained from $G(A)$ by multiplying every arc label by r. The usual properties of scalar multiplication are also easily verified.

Given an $m \times n$ matrix A and an $n \times r$ matrix B, the **concatenation graph** $G(A) * G(B)$ is defined by taking the digraphs $G(A)$ and $G(B)$ and identifying the n column vertices of $G(A)$ with the n row vertices of $G(B)$. An example of the concatenation of two graphs is given in FIGURE 2.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 4 & -1 \\ 0 & -2 \end{pmatrix}$$

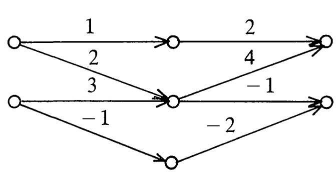

FIGURE 2. **The concatenation of two König digraphs.**

THEOREM 1. *The (i, j) entry of the matrix product AB is equal to the sum of the weights of the paths in $G(A) * G(B)$ from the ith row vertex of $G(A)$ to the jth column vertex of $G(B)$.*

Proof. Each path of length two from vertex i to vertex j passes through a unique vertex k. By definition, the weight of the path is the product of the labels of its arcs, which is $a_{ik}b_{kj}$. Summing over all possible k gives the desired result.

Note that the exclusion of edges of weight zero from the digraph according to our convention still leaves the theorem valid. We see, for example, that for the digraph in FIGURE 2, the (1,2) entry of the product AB is -2 since there is only one path between the appropriate vertices.

Several properties of matrix multiplication are immediate consequences of Theorem 1. To show *matrix multiplication is associative*, assume that matrices A, B, and C are of appropriate orders, and consider the graph $G(A) * G(B) * G(C)$. Then the (i, j) entry of $(AB)C$ and $A(BC)$ are clearly both equal to the sum of the weights of the paths of length three from the ith row vertex of $G(A)$ to the jth column vertex of $G(C)$. It is an easy exercise to show that *matrix multiplication distributes over addition* by looking at a particular path (i, j, k) in the König digraphs of $G(A) * G(B)$, $G(B) * G(C)$, and $G(A) * G(B + C)$.

If P is a permutation matrix, then $G(P)$ consists of a set of arcs with no common vertices. In other words, the outdegree of each row vertex and the indegree of each column vertex is equal to one, and all labels of arcs are 1. *The product of permutation matrices is a permutation matrix.* Look at the concatenation digraph! Likewise, the digraph of a diagonal matrix makes the following facts trivial. *The product of diagonal matrices is a diagonal matrix. Each diagonal entry of the product is the product of the corresponding diagonal entries in the factors.*

A matrix is upper (lower) triangular if $a_{ij} = 0$ whenever $i > j$ $(i < j)$. Thus $G(A)$ is the König digraph of an upper (lower) triangular matrix if and only if all the arcs in the drawing are horizontal or downward (upward). *The product of upper (lower) triangular matrices is upper (lower) triangular.* The digraph makes it trivial!

The König digraph of the transpose of A is obtained from $G(A)$ by reversing the direction of all of the arcs. If we wish to adhere to the convention of having row vertices on the left and column vertices on the right, we must then reflect the new graph with respect to a line through the column vertices (see FIGURE 3). The properties of the reversals and reflection make the following Theorem obvious.

THEOREM 2. *Let A^T be the transpose of the matrix A; then*

$$(1)\quad (A^T)^T = A,$$
$$(2)\quad (A + B)^T = A^T + B^T,$$
$$(3)\quad (cA)^T = cA^T, \text{ and}$$
$$(4)\quad (AB)^T = B^T A^T.$$

Notice how the reversal of the arcs makes the reversal of the order of the matrices A and B in Equation (4) intuitively clear.

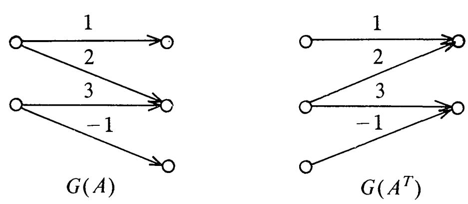

FIGURE 3

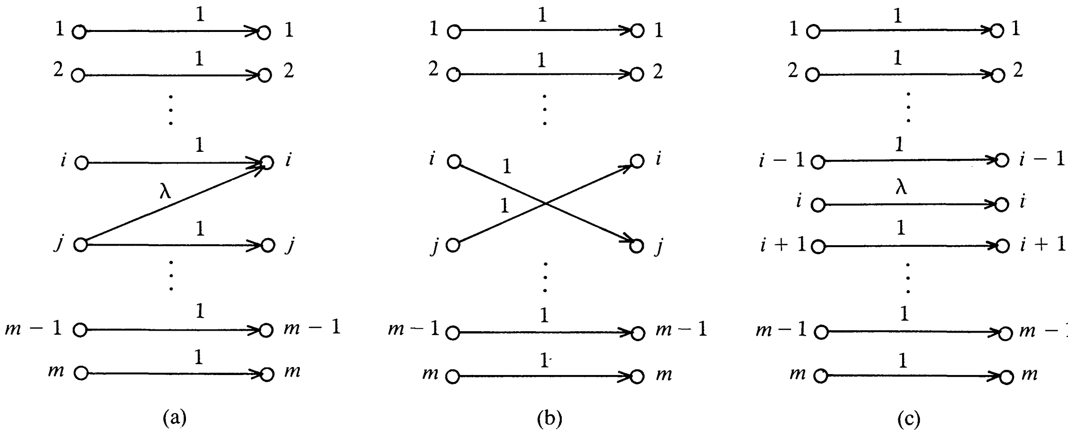

FIGURE 4. König digraphs of the elementary row operations on an $m \times n$ matrix: (a) adds a multiple of the ith row to the jth row; (b) interchanges the ith and jth rows; (c) multiplies the ith row by λ.

The (i, j) element of the product matrix AB is completely determined by the subgraph of $G(A) * G(B)$ consisting of the ith row vertex of $G(A)$, the jth column vertex of $G(B)$, and all paths of length two joining them. A partition of the column vertices of $G(A)$ (= the row vertices of $G(B)$) produces a partition of these paths. The validity of block multiplication is now easy to understand since each block arises from a partition of the rows and columns of the matrix.

Properties of matrices associated with elementary row operations are easily seen using König digraphs, since multiplying by a matrix on the left is nothing more than concatenation on the left by a König digraph. For example, if we wish to multiply the ith row of the $m \times n$ matrix A by λ and add it to the jth row, we simply concatenate the digraph shown in FIGURE 4(a) with $G(A)$. It is an easy exercise to construct the König digraphs that represent the interchange of two rows of a matrix or the multiplication of a row by λ. See FIGURE 4(b), (c). The digraphs in FIGURE 4 are called the König digraphs of the elementary row operations. A glance at these quickly shows that the König digraphs of an elementary row operation and its inverse are identical except for at most one label; in one case λ is replaced by $-\lambda$ and in the other it is replaced by λ^{-1}.

The Coates graph and the determinant

In this section we look at a different digraph associated with a square matrix. The determinant of the matrix can be computed from the weights of the cycles in this graph. This allows a relatively straightforward computation of the determinant of a matrix of arbitrary order, especially if there are many zero entries. Usually determinants are defined by making an excursion into the theory of permutations, a subject which by its nature is deeper than the determinant concept itself and necessitates a relatively difficult digression. An alternative approach is to define the determinant inductively using the cofactors, but like many inductive approaches, the results of the proofs are often believed but not really understood. The graph-theoretic approach avoids both of these pitfalls. The proof, for example, that the determinant of a matrix A and the determinant of A^T are equal tends to become lost in notation when using either the permutation or inductive approach. With the graph-theoretic approach this result is proven in one sentence!

Recall that a factor F of a digraph H is a subgraph containing all the vertices of H in which each vertex has both indegree and outdegree equal to one. In other words, it consists of a collection of disjoint cycles that go through each vertex of H. The number of cycles in the factor F is denoted $n(F)$. If the digraph H is labelled, then $W(F)$ denotes the weight of the factor.

Given a square matrix $A = (a_{ij})$ of order n, the **Coates digraph** $D(A)$ is a labelled digraph with n vertices where the arc from vertex i to vertex j has a_{ij} as a label. In FIGURE 5, the Coates

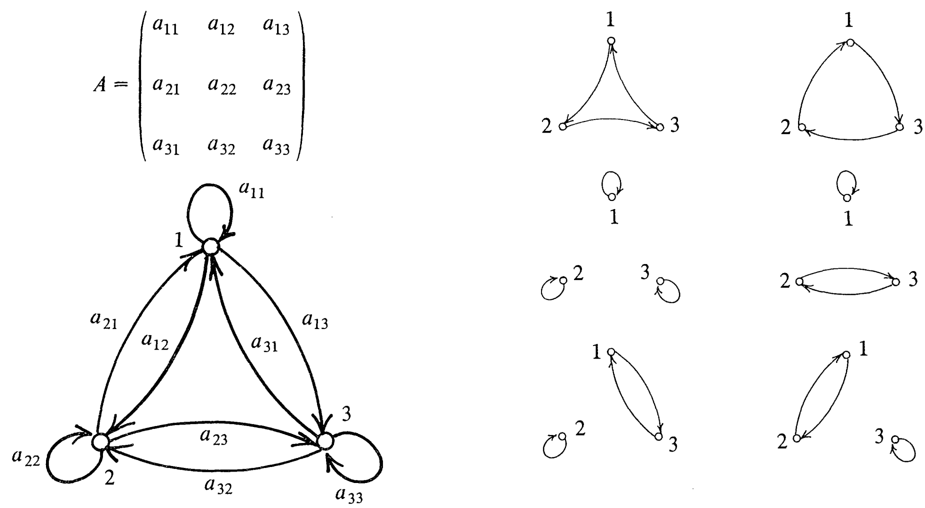

FIGURE 5. **The Coates digraph $D(A)$ of a matrix A of order $n=3$, and the six factors of $D(A)$.**

digraph $D(A)$ of a 3×3 matrix A is shown. The digraph $D(A)$ for $n=3$ has six factors; the two having no loops are the cycles $(1,2,3,1)$ and $(1,3,2,1)$, the three with one loop are $(1,1)$ $(2,3,2)$, $(2,2)$ $(1,3,1)$ and $(3,3)$ $(1,2,1)$; finally, $(1,1)$ $(2,2)$ $(3,3)$ consists of three loops.

Given a square matrix A of order n, $D(A)$ its Coates digraph, and $\mathbf{F}$ the set of all factors of $D(A)$, then the determinant of A is defined

$$\det A = (-1)^n \sum_{F\in \mathbf{F}} (-1)^{n(F)} W(F). \tag{1}$$

This formulation of the determinant first appeared in 1959 [**1**], although some of the relationships between graphs and determinants were known much earlier [4]. Except for the work of Cvetković [**2**], it has received attention only in a few scattered research papers. In the example of the 3×3 matrix (FIGURE 5), the factors having no loops contribute $-a_{12}a_{23}a_{31}$ and $-a_{13}a_{32}a_{21}$ to the sum, those having one loop contribute $a_{11}a_{23}a_{32}$, $a_{13}a_{22}a_{31}$, and $a_{12}a_{21}a_{33}$ to the sum, and the final factor adds $-a_{11}a_{22}a_{33}$. Since $(-1)^3=-1$, we observe that equation (1) yields the standard result. It is straightforward to verify the validity of formula (1) for the determinant for $n=1$, $n=2$, and $n=4$. For $n=4$, we note that the set of all factors can be partitioned into 9 factors with no loops, 8 factors with one loop, 6 factors with two loops, and one factor with 4 loops.

We now prove that formulation (1) of the determinant is identical with the usual permutation definition.

THEOREM 3. *The graphic and permutation definitions of* $\det A$ *are equivalent, i.e., if* $\det A$ *is defined by equation* (1), *then*

$$\det A = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

where S_n is the set of all permutations of the integers $\{1,2,\dots,n\}$.

Proof. For any permutation σ, the set of arcs $\{(i,\sigma(i))|i=1,\dots,n\}$ forms a factor F. Conversely, each factor F also corresponds to a permutation σ. Hence to complete the proof we need only show that the two definitions produce values with the same sign, i.e., that $(-1)^n(-1)^{n(F)}=\operatorname{sgn}(\sigma)$. The vertices in a cycle of the factor F correspond precisely to the integers in a cycle in the decomposition of the corresponding permutation σ. In addition, a cyclic permutation is even if and only if the corresponding cycle in the digraph has an odd number of vertices. Let $e(\sigma)$ and $o(\sigma)$ denote the number of even and odd cycles in the permutation σ (or, equivalently, the odd and even cycles in the factor). Then

$$\operatorname{sgn}(\sigma)=(-1)^{o(\sigma)},\ n(F)=e(\sigma)+o(\sigma),$$

and n and $e(\sigma)$ have the same parity. Hence

$$\operatorname{sgn}(\sigma)=(-1)^{o(\sigma)}=(-1)^{n(F)-e(\sigma)}=(-1)^{e(\sigma)}(-1)^{n(F)}=(-1)^{n}(-1)^{n(F)}.$$

Using definition (1), here is the proof that *the determinant of a matrix and the determinant of its transpose are equal*. The Coates digraph $D(A^T)$ is obtained from $D(A)$ by reversing the orientation of each arc; this leaves the factors, the weight of each factor, and the number of cycles in each factor unchanged, and hence the determinant is unchanged.

In the last section, we considered matrices that correspond to the elementary row operations. The matrix that interchanges two rows will have Coates digraph of a single factor, i.e., $n-2$ loops and one cycle of length two. Since all weights are equal to one, the determinant is $(-1)^n(-1)^{n-1} = -1$. Adding a multiple of one row to another also corresponds to a Coates digraph with only one factor, and the determinant is 1 in that case. Finally, multiplying a row by λ corresponds to a diagonal matrix, and this matrix has a Coates digraph with only loops, hence its determinant is λ.

Consider an upper triangular matrix as a further example. As can be seen in FIGURE 6, if the vertices of its Coates digraph are placed horizontally in increasing order, then all arcs are loops or go from left to right, and hence the only possible cycles are loops. But this means that the only (nonzero) factor is (1,1) (2,2) (3,3)...(n,n), hence

$$\det A=(-1)^{n}(-1)^{n}a_{11}a_{22}\cdots a_{nn}.$$

We have proved *the determinant of an upper triangular matrix is the product of its diagonal elements*.

A slightly harder combinatorial result can be obtained if we let $C(n,r)$ denote the number of combinations of n things taken r at a time, and let A be a cyclic tridiagonal matrix, i.e., the nonzero entries of A satisfy $a_{ii}=d$, $a_{i\,i+1}=c$, or $a_{i+1\,i}=b$. The reader can prove that

$$\det A=\sum_{r=0}^{[n/2]}C(n-r,r)b^rc^rd^{n-2r}.$$

Now let us look at the effect of elementary row operations on the Coates digraph and consequently on the determinant.

THEOREM 4. *Let A be a square matrix of order n, $1\leqslant i,j\leqslant n$, and let*
B be obtained from A by multiplying the ith row by λ,
C be obtained from A by interchanging the ith and jth rows,
E be obtained from A by adding the jth row to the ith row.

Then $\det B=\lambda\det A$,
$\det C=-\det A$, *and*
$\det E=\det A$.

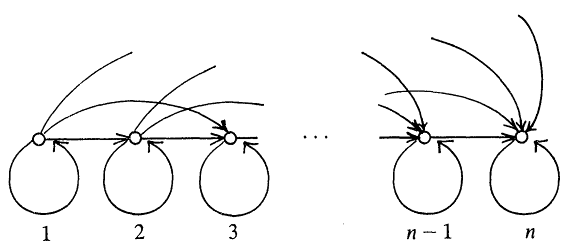

FIGURE 6. **The Coates graph of an upper triangular matrix.**

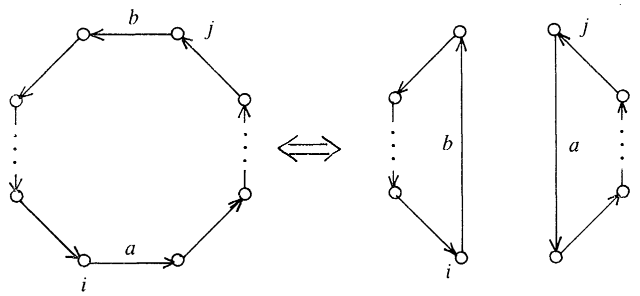

FIGURE 7. **The change in a factor under row interchange.**

Proof. $D(B)$ is obtained from $D(A)$ by multiplying the label of each arc going out of the ith vertex by λ. Thus F is a factor of $D(A)$ if and only if F is a factor of $D(B)$. Since the outdegree of each vertex in F is one, the weights of all arcs of F in $D(A)$ but one are the same as the weights of F in $D(B)$, the exception being that single arc going out of the ith vertex which has been multiplied by λ. Thus the weight of F in $D(B)$ is λ times that in $D(A)$, and, summing over all of the factors, $\det B = \lambda \det A$.

The Coates graphs $D(A)$ and $D(C)$ can be related as follows: we wish to take the digraph $D(A)$ and move some of the arcs. Since $a_{ik} = c_{jk}$ and $a_{jk} = c_{ik}$, $D(C)$ is obtained from $D(A)$ by taking each arc going out of the ith vertex and moving it so that it goes out of the jth vertex, and conversely, moving the arcs going out of the jth vertex so that they can then go out of the ith vertex (the terminal end of the arc remains fixed). The net change of these movements of arcs is to interchange a_{ik} and a_{jk} for every k, which, of course, is just what is desired. Now consider a particular factor F, and suppose that the labels of the arcs going out of vertex i and vertex j are a and b. How do these row interchanges affect F? It remains a factor with $W(F)$ unchanged! As can be seen in FIGURE 7, if both the ith and the jth vertices are in the same cycle to begin with, then the movement of the arcs splits the cycle into two new ones with the same weight (left to right in FIGURE 7). On the other hand, if the ith and jth vertices are in different cycles, then the movement of the arcs causes them to combine into one cycle with the same weight (right to left in FIGURE 7). In either case, $W(F)$ is unchanged and $n(F)$ is increased or decreased by one. Thus the sign of each summand changes, and $\det C = -\det A$.

Notice that if A has two identical rows, then interchanging them leaves A unchanged but reverses the sign of $\det A$, and so $\det A = 0$. Now let A' be obtained from A by replacing the ith row of A by the jth row of A (so that $\det A' = 0$). Consider a factor F of $D(E)$; it is also a factor of $D(A)$ and $D(A')$, and all the weights of the arcs are identical except for the arc in F going out of the ith vertex. The weight of this arc is $a_{ik} + a_{jk}$ in E, is a_{ik} in A, and is a_{jk} in A'. Thus the weight of F in $D(E)$ is the sum of the weight of F in $D(A)$ and the weight of F in $D(A')$. Summing over all factors F, we get $\det E = \det A + \det A' = \det A$.

If one already knows how to reduce a matrix to reduced row echelon form, then it is easy to see from Theorem 4 that $\det EA = \det E \cdot \det A$ for an elementary matrix E and consequently that $\det AB = \det A \cdot \det B$ holds in general. In the next section we shall prove this result with graph-theoretic tools.

Sometimes the determinant is defined abstractly as a function from the set of square matrices of order n to the real numbers which, viewed as an n-variable function of the columns, is alternating, multilinear, and takes the identity matrix to 1. Since $\det A^T = \det A$, we can use rows instead of columns, of course. Suppose F is a factor of a directed graph with n vertices. Then, by the argument in Theorem 4, the mapping that takes a square matrix M into $(-1)^n W(F)$ is a linear function in each row of M. Hence, summing over all rows, we see from (1) that the determinant is indeed an alternating multilinear form on the rows of M, and that the determinant of the identity matrix is 1. Since it is easily seen that such a form is unique (see [**6**] p. 191, for example), we now see that the graphic definition is equivalent to the abstract definition.

We now consider expansion of the determinant by cofactors. Given a square matrix A, let D_{ij} be defined as the determinant of the matrix obtained by deleting the ith row and jth column from A. The **(i,j) cofactor of A**, denoted A_{ij}, is then defined by

$$A_{ij} = (-1)^{i+j} D_{ij}.$$

Now consider the set $\mathbf{F}$ of all factors of $D(A)$ and a particular vertex i. Each factor contains a unique arc going out of the ith vertex. Let $\mathbf{F}_j$ be the set of factors containing the arc (i, j). Then clearly $\mathbf{F}$ is partitioned by $\mathbf{F}_j, j = 1,2,\dots,n$. The (i, j) cofactor can now be expressed in terms of $\mathbf{F}_j$.

THEOREM 5. *Let $A = (a_{ij})$ be a square matrix of order n, $1 \leqslant j \leqslant n$, and let $\mathbf{F}_j$ be the set of factors of $D(A)$ containing the arc (i, j). Then*

$$a_{ij} A_{ij} = (-1)^n \sum_{F \in \mathbf{F}_j} (-1)^{n(F)} W(F). \tag{2}$$

Proof. First, suppose that $i = j$, and let A' be the square matrix of order $n-1$ obtained by deleting the ith row and column from A. By definition, any F in $\mathbf{F}_i$ will contain the loop (i, i). Thus the factors in $\mathbf{F}_i$ are precisely the factors of $D(A')$ plus the loop (i, i). Since the weight of a factor in $\mathbf{F}_i$ is the product of a_{ii} and the weight of a factor in $D(A')$, the result is clear except, possibly, for the sign. The factor in $\mathbf{F}_i$ has one more cycle than the corresponding factor in $D(A')$, so an extra multiple of -1 is introduced inside the summation. However the order of $D(A)$ is one more than that of $D(A')$, and so one fewer multiple of -1 appears outside the summation; thus the theorem is valid when $i = j$.

To complete the proof of the theorem, we must see the effect of interchanging two adjacent columns of A on the left and right side of the equation (2). Let A' be obtained from A by interchanging columns r and $r+1$. We certainly have $a_{ir} = a'_{i\,r+1}$ and

$$A_{ir} = (-1)^{i+r} D_{ir} = -\left((-1)^{i+r+1} D'_{i\,r+1}\right).$$

Thus interchanging two adjacent columns causes the value of the left side of the equation (2) to change only in sign. For the right side, we proceed as in the proof of Theorem 4 and rearrange the arcs of $D(A)$ to form $D(A')$. Since we are interchanging columns, we move the terminal end rather than the initial end of the arcs. Thus each arc terminating at the rth vertex is moved so it terminates at the $r+1$st vertex and vice-versa. In a manner analogous to that illustrated in FIGURE 7, each factor F in $\mathbf{F}_r$ becomes a factor F' in $\mathbf{F}_{r+1}$ with $W(F) = W(F')$ and $|n(F) - n(F')| = 1$. Summing over all F in $\mathbf{F}_r$, we get

$$(-1)^n \sum_{F \in \mathbf{F}_r} (-1)^{n(F)} W(F) = -\left((-1)^n \sum_{F' \in \mathbf{F}_{r+1}} (-1)^{n(F')} W(F')\right).$$

Thus the right side of equation (2) also changes sign when two adjacent columns are interchanged. Hence equation (2) is valid for A if and only if it is valid for A'. In other words, if the theorem is valid for a particular i and j, then it is also valid for i and $j+1$.

Now suppose $i > j$; we then successively interchange columns j and $j+1$, $j+1$ and $j+2$, $j+2$ and $j+3$, etc., until we have finally interchanged columns $i-1$ and i. By the preceding argument, all of the resulting matrices will simultaneously satisfy or not satisfy equation (2) of the theorem. Since the final matrix is just the case where $i = j$, we see that the theorem is valid for all of the matrices, and, in particular, for A. The argument for $i < j$ is symmetric.

COROLLARY (*Expansion by cofactors*). *Let A be a square matrix of order n, $1 \leqslant i, j \leqslant n$; then*

$$\det A = \sum_{k=1}^{n} a_{ik} A_{ik} = \sum_{k=1}^{n} a_{kj} A_{kj}.$$

Proof. As was noted previously, $\mathbf{F}$ is partitioned by $\mathbf{F}_k$, $k = 1,2,\dots,n$. Thus

$$\det A = (-1)^n \sum_{F \in \mathbf{F}} (-1)^{n(F)} W(F) = \sum_{k=1}^{n} (-1)^n \sum_{F \in \mathbf{F}_k} (-1)^{n(F)} W(F) = \sum_{k=1}^{n} a_{ik} A_{ik}.$$

The Coates digraph has application in other areas of linear algebra including powers of matrices and spectral theory, although these applications generally focus on the paths in the digraph rather than on the cycles. For the sake of brevity, these applications must be omitted, but even the results presented here establish the natural connection between the Coates digraph and linear algebra.

Another view of the determinant

We now wish to return to the determinant of the product of two matrices. Our purpose is to give a graph-theoretic proof, not only as an end in itself, but also as a means of revealing some of the underlying combinatorial structure. For a square matrix of order n, we must first observe how a factor in the Coates digraph translates to the König digraph. Such a factor is simply a vertex-disjoint set of n arcs, and so, disregarding the signature for the moment, the summands of the determinant (Equation (1)) are precisely the weights of the subgraphs consisting of n vertex-disjoint arcs.

Now consider the concatenation of the König digraphs of two square matrices A and B of order n. How does this relate to the product of $\det A$ and $\det B$? If $\mathbf{F}_1$ is the set of factors of $G(A)$ and $\mathbf{F}_2$ is the set of factors of $G(B)$, then

$$\det A \cdot \det B = \left(\sum_{F_1 \in \mathbf{F}_1} (-1)^{n_1} W(F_1) \right) \left(\sum_{F_2 \in \mathbf{F}_2} (-1)^{n_2} W(F_2) \right)$$
$$= \sum_{\substack{F_1 \in \mathbf{F}_1 \\ F_2 \in \mathbf{F}_2}} (-1)^{n_1 + n_2} W(F_1) W(F_2).$$

In other words, each summand corresponds to the weight of a subgraph in $G(A) * G(B)$ consisting of n vertex-disjoint paths of length 2. Disregarding the sign for the moment, we see that these subgraphs in fact correspond precisely to the summands of $\det A \cdot \det B$. What about the summands of $\det AB$? Each arc in AB is obtained from the paths of length 2 in $G(A) * G(B)$. Further, each arc of length 2 in $G(A) * G(B)$ will contribute to some element in AB and hence to $\det AB$. Again, expanding the products of sums we see that the summands of $\det AB$ are the weights of subgraphs of $G(A) * G(B)$ consisting of n paths of length 2 where the initial and terminal (but not necessarily the middle) vertices are distinct. The non-distinctness of the middle vertex accounts for the larger number of terms ($n!n^n$) in $\det AB$ as compared with the number $(n!)^2$ in $\det A \cdot \det B$.

To complete the proof we return to the postponed consideration of the signs of the terms. We shall show that each of the summands in $\det A \cdot \det B$ also appears in $\det AB$ with the same sign and that the remaining terms of $\det AB$ sum to zero. To do this we must first see a new relationship between the parity of a permutation σ and the König digraph. Let the König digraph of a permutation σ be the digraph of the corresponding permutation matrix, so the König digraph of σ simply consists of the set of arcs $(i, \sigma(i))$.

THEOREM 6. *The parity of a permutation σ and the parity of the number of arc intersections in the König digraph are the same.*

Proof. Suppose $\sigma(i) = i$ for $i = 1, 2, \ldots, n$. Then the number of intersections in the König digraph of σ is 0 and σ has even parity. If σ is an arbitrary permutation, we may proceed by uncrossing the arcs one pair at a time, while observing that the parity of both the permutation and the number of arc intersections change with each uncrossing.

Indeed, if the arcs $(i, \sigma(i))$ and $(j, \sigma(j))$ intersect, we may say without loss of generality that $i < j$ and $\sigma(i) > \sigma(j)$. Define σ' by $\sigma'(i) = \sigma(j)$, $\sigma'(j) = \sigma(i)$, and $\sigma'(k) = \sigma(k)$ for all other k. How does the number of arc intersections of σ' compare with that of σ? Any k not equal to i or j

must satisfy $1 \leqslant k < i$ or $i < k < j$, or $j < k \leqslant n$. Similarly, $1 \leqslant \sigma(k) < \sigma(j)$, $\sigma(j) < \sigma(k) < \sigma(i)$, or $\sigma(i) < \sigma(k) \leqslant n$. Thus there are nine possible configurations for k and $\sigma(k)$. For eight of them, the number of arc intersections is unchanged by uncrossing the arcs $(i, \sigma(i))$ and $(j, \sigma(j))$. In the configuration where $i < k < j$ and $\sigma(j) < \sigma(k) < \sigma(i)$ the number of intersections drops by two. Thus the total number of arc intersections has been decreased by twice the number of arcs of the form $\{(k, \sigma(k)) | i < k < j, \sigma(j) < \sigma(k) < \sigma(i)\}$ plus one, the extra intersection coming from the arcs $(i, \sigma(i))$ and $(j, \sigma(j))$ themselves. Hence the transposition (ij) that takes σ to σ' also causes a change in the parity of the number of arc intersections. Eventually, we get to the identity permutation, and hence the result is established.

COROLLARY. *The permutation σ and the number of pairs in the set $\{(i, j) | 1 \leqslant i < j \leqslant n, 1 \leqslant \sigma(j) < \sigma(i) \leqslant n\}$ have the same parity.*

The set defined in the Corollary is called the set of **inversions** of the permutation σ.

Note that in the proof of Theorem 6, each uncrossing represents a transposition, and the sequence of uncrossings that are used to obtain the identity permutation yields the product of transpositions that equals the original permutation. Thus the proof also shows that *any permutation is the product of transpositions*, and the graph yields an intuitive representation of σ as the product of transpositions.

Now let us apply these results to the products of determinants. One summand of $\det A \cdot \det B$ corresponds to n vertex-disjoint arcs of length two. The sign of this term is the product of the signs of the terms in $\det A$ and in $\det B$, each of which can be obtained from the number of arc intersections in its particular case. How does this compare with the same term as it appears in $\det AB$? The sign of this term comes from the parity of the number of intersections of the paths of length two. A pair of paths will intersect if the first arcs of the paths intersect in $G(A)$ or if the second arcs intersect in $G(B)$. But notice that if both the first arcs and second arcs intersect, then they yield a pair of non-intersecting arcs as far as $\det AB$ is concerned. Thus the number of intersecting arcs in $\det AB$ and the product of those in $\det A$ and $\det B$ have the same parity, which is what was desired.

Finally, let us take care of those extra terms that were in $\det AB$ but did not appear in $\det A \cdot \det B$. Let (i, k, l) and (j, k, m) be two paths of length two in $G(A) * G(B)$. How can these paths appear in $\det AB$? There are two ways: they come from terms of the form $(a_{ik} b_{kl})(a_{jk} b_{km})x$ or terms of the form $(a_{ik} b_{km})(a_{jk} b_{kl})x$ where x represents the remaining terms in each of the products. Hence the permutations that give rise to these paths can be paired so that they differ only by an interchange of l and m. Thus the same product appears with opposite sign, and they sum to zero.

It is interesting to note that if one considers more than two paths passing through the vertex k, then the fact that the alternating group contains exactly half of the members of the symmetric group can yield the same result.

The author would particularly welcome comments from readers. Further material can be supplied for those with such interest.

This paper results in part from joint work of the author and his friend and colleague Dragoš Cvetković of the University of Belgrade, whose textbook [3] presents linear algebra from a graph-theoretic viewpoint.

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NOTES

Arithmetic in Complex Bases

WILLIAM J. GILBERT
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

In this note, we introduce a novel way of doing complex arithmetic that does not involve separating the complex numbers into their real and imaginary parts. This method uses the representation of complex numbers in positional notation using a complex base $-n+i$, for a positive integer n, with natural numbers as digits. Addition, subtraction and multiplication can be performed directly in this positional notation and is similar to real decimal arithmetic; the main difference is in the carry digits. However, division is more complicated and the construction of a good algorithm for long division is a challenging unsolved problem.

We say that an integer z (real or complex) is **represented in the base** b with **digits** from the set $\mathcal{D}$ if it is written in the form

$$z=a_k b^k+a_{k-1}b^{k-1}+\cdots+a_1 b+a_0,$$

where $a_k, a_{k-1},\ldots,a_1,a_0\in\mathcal{D}$. We denote this representation of z by the **positional notation** $(a_k a_{k-1}\ldots a_1 a_0)_b$. It is well known that the natural numbers can be represented in any integral base $b>1$ using the digit set $\mathcal{D}=\{0,1,2,\ldots,b-1\}$ and arithmetic can be performed in any of these bases; of course, the decimal (base 10) and binary (base 2) representations are the most popular. All the real integers, both positive and negative, can be uniquely represented by means of a negative integral base $-b<-1$ using the natural number digit set $\mathcal{D}=\{0,1,2,\ldots,b-1\}$. Reference [**4**] contains details of the arithmetic in these negative bases.

The integers in the field of complex numbers, called **Gaussian integers**, are of the form $x+iy$, where x and y are real integers. For each fixed positive integer n, Kátai and Szabó [**5**] proved that *all the Gaussian integers can be uniquely represented in the base* $b=-n+i$ *using the digit set* $\mathcal{D}=\{0,1,2,\ldots,n^2\}$. They also showed that these bases and their conjugates are the only possible ones in which the digit set consists of the natural numbers $0,1,2,\ldots,\ \mathrm{Norm}(b)-1$.

The base $b=-1+i$ provides a binary representation of the complex numbers using 0 and 1 as digits; for example

$$(1011)_{-1+i}=(-1+i)^3+(-1+i)+1=2+3i,$$
$$(1100)_{-1+i}=(-1+i)^3+(-1+i)^2=2.$$

The base $b=-3+i$ yields a decimal representation using the digit set $\mathcal{D}=\{0,1,2,\ldots,9\}$; for example, $5+6i$ is written in positional notation as $(1443)_{-3+i}$. An efficient method for converting a number into a complex base will be given later. See [**6**, § 4.1] for further details of the history of negative and complex bases.

Addition and multiplication of two numbers written in positional notation in base $-n+i$ can be performed in the same way as real arithmetic in base n^2+1, except for a change in the carry digits. The allowable digits in base $-n+i$ are $0,1,2,\ldots,n^2$, so whenever the sum of one column exceeds n^2, then n^2+1, or some multiple of it, has to be carried to the higher columns. Since $n^2+1=(1\ 2n-1\ (n-1)^2\ 0)_{-n+i}$, an overflow of n^2+1 in one column means that the digits $1\ 2n-1\ (n-1)^2$ have to be carried to the next *three* higher columns. The following examples illustrate some of this arithmetic. For clarity the subscripts for each base have been omitted in the displayed calculations, and the Cartesian form $z = x + iy$ of the numbers is shown alongside the complex base calculation. The carry digits are placed beneath the sum. There are various ways to subtract in the base $-n + i$; one method is to multiply the subtrahend by negative one and then add.

EXAMPLE 1. The calculations below illustrate addition and multiplication of $2 + 3i$ and $-1 - i$ in base $-1 + i$.

```
    1011      2 + 3i          1011        2 + 3i
  +  110     -1  - i        ×  110      ×  -1  - i
 -------     -------     ----------     ----------
 1110101      1 + 2i         10110
     110                    101100
   110                 11101001010         1 - 5i
                               110
                              110
                             110
                            110
```

EXAMPLE 2. The calculations below give an arithmetical check that $i^2 + 1 = 0$ in base $-3 + i$.

```
   13        i            169       i^2
  ×13      × i          +   1     +  1
  ---      ---        -------     ----
  130                ...0000000      0
   39                      154
  ---                     154
  169      i^2           154
                        154
                       154
                      154
                      . .
```

The addition shown in Example 2 illustrates a problem that arises in negative and complex bases. This is the fact that there can be an infinite series of carry digits, even though the sum is finite. This phenomenon must always happen whenever a number and its negative can be represented in the same base using natural numbers as digits. (See also [4].) This infinite sequence of carry digits does not invalidate the arithmetic because the carry numbers all sum to zero after a certain stage. In the above example, the numbers inside the dotted triangle sum to zero.

It can be proved that the number of digits in the sum of two numbers expressed in the base $-n + i$ is at most three more than the number in the largest summand if $n \geqslant 4$; at most five more if $n = 2$ or 3 and at most eight more if $n = 1$. The following examples show the extreme cases in the bases $-1 + i$ and $-3 + i$.

EXAMPLE 3. The calculations below show additions with long totals in bases $-1 + i$ and $-3 + i$ respectively.

```
          1011     2 + 3i         905     77 - 54i
  +       1011     2 + 3i     +   655     38 - 31i
 -------------    -------    --------    ---------
  111010010100     4 + 6i    15609090    115 - 85i
           110                    154
          110                   154
         110                    154
        110                   154
       110
      110
     110
```

This addition could be automated to add two k-digit numbers in base $-3+i$ and to stop after $k+5$ digits. The sum obtained would be correct regardless of whether any carry digits remain; the extra digits must sum to zero.

The propagation of carry digits in real arithmetic has been a concern of computer scientists who are trying to speed up arithmetical operations. This problem is accentuated in complex base arithmetic. One suggested solution for real arithmetic [1] extends very neatly to complex bases and avoids infinite carries. Each representation of an integer in the base b can be viewed as a polynomial in b. The arithmetical operations of addition, subtraction and multiplication can be first done, without carries, in the ring of formal polynomials $\mathbb{Z}[b]$ by permitting the coefficients of the powers of b to be any integer, not just the allowable digits for the base. At the end of all the calculations the resulting polynomial can be "cleared," using the minimum polynomial of b, so that all the coefficients are digits lying in the correct range. In the examples which follow, a polynomial $a_k b^k + \cdots + a_1 b + a_0 \in \mathbb{Z}[b]$ will be written in positional notation as $(a_k \ldots a_1 a_0)$. The minimum polynomial of $-n+i$ is $b^2+2nb+n^2+1$ or $(1\ 2n\ n^2+1)$ in base b positional notation; adding any multiple of this minimum polynomial to any other polynomial will not affect its value when $b=-n+i$ because $b^2+2nb+n^2+1=0$. The **clearing algorithm** is as follows. Let a_r be the coefficient of the smallest power of b which lies outside the range from 0 to n^2. Then there exists an integer s such that $0 \leqslant a_r + s(n^2+1) \leqslant n^2$. Add sb^r times the minimum polynomial to clear this rth coefficient. The following example shows this algorithm in operation using the base $-3+i$ whose minimum polynomial is (1 6 10). Because the *minimum* polynomial of the base is used in this process, the algorithm will terminate after a finite number of steps [3] and hence this method avoids any infinite series of carries.

EXAMPLE 4. The following calculations show the multiplication $(182)_{-3+i} \times (38)_{-3+i} = (13546)_{-3+i}$ using the clearing algorithm.

$$
\begin{array}{rrrrrl}
 & & 1 & 8 & 2 & \\
\times & & & 3 & 8 & \\
\hline
 & & 8 & 64 & 16 & \\
 & 3 & 24 & 6 & & \text{polynomial multiplication} \\
\hline
 & 3 & 32 & 70 & 16 & \\
\hline
 & & -1 & -6 & -10 & \\
 & -6 & -36 & -60 & & \text{clearing algorithm} \\
1 & 6 & 10 & & & \\
\hline
1 & 3 & 5 & 4 & 6 &
\end{array}
\qquad
\begin{array}{r}
-14+2i \\
\times -1+3i \\
\hline
\\ \\ \\ \\ \\ \\
=8-44i \\
\hline
\end{array}
$$

The easiest way to convert any Gaussian integer into a complex base is to use this clearing algorithm. A Gaussian integer $s+it$ can be written as $t(-n+i)+(s+nt)$ or as $(t\ s+nt)$ in positional notation in base $-n+i$. The clearing algorithm will now convert this to a representation in base $-n+i$ with the digits in the proper range from 0 to n^2.

EXAMPLE 5. The following calculations illustrate the conversion of $5+6i$ into its base $-3+i$ representation, $(1443)_{-3+i}$.

$$
\begin{array}{rrrrl}
 & & 6 & 23 & = 5+6i \\
 & -2 & -12 & -20 & \\
1 & 6 & 10 & & \\
\hline
1 & 4 & 4 & 3 & \\
\hline
\end{array}
$$

The positional representation of the Gaussian integers in base $b=-n+i$ can be extended to cover all the complex numbers by using infinite radix expansions. Each complex number can be written as a convergent sum

$$\sum_{j=-\infty}^{k} a_j b^j, \qquad (1)$$

where the digits $a_j \in \{0, 1, 2, \ldots, n^2\}$. The expansion (1) is written in positional notation, using a radix point, as $(a_k a_{k-1} \ldots a_0 . a_{-1} a_{-2} \ldots)_b$. For example,

$$(5+i)/4 = 1 + (-1+i)^{-2} + (-1+i)^{-3} = (1.011)_{-1+i},$$

$$\sqrt{2} + i = (15.49778016\ldots)_{-3+i},$$

and

$$1/3 = (1.\overline{4724})_{-3+i},$$

where the bar over a string of digits indicates that they are to be repeated indefinitely.

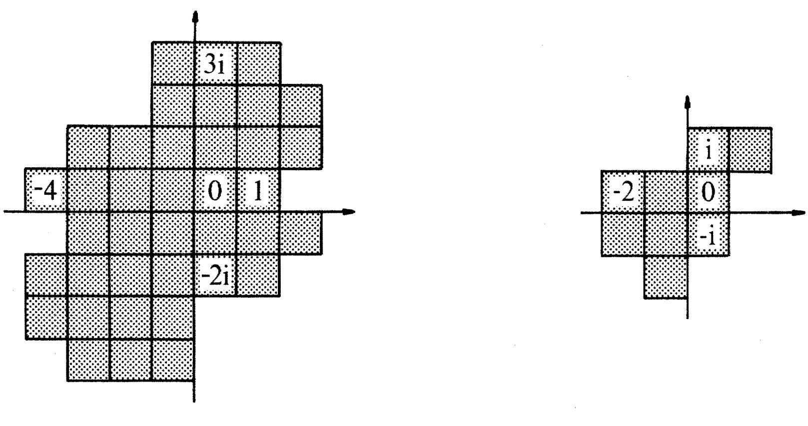
As in real systems, complex numbers of the form $x + iy$, with x and y rational, have periodic or terminating expansions in base $-n+i$. All the other complex numbers have aperiodic expansions. Also, as in real systems, these expansions are not always unique. In complex bases some numbers have one expansion, some two, and a few even have three different expansions. For example, $(1-2i)/5 = (0.\overline{001})_{-1+i} = (1.\overline{100})_{-1+i} = (111.\overline{010})_{-1+i}$. Reference [2] discusses the geometric significance of the points with multiple expansions.

Long division in real arithmetic consists of dividing one finite expansion by another. By shifting the radix point of the divisor and dividend this is equivalent to dividing one integer by another. The long division algorithm in real arithmetic for dividing one natural number c by d in the positive base b is as follows. Initially set $c = ad + r_0$, where $0 \leqslant r_0 < d$, and a is an integer. Then, for $j > 0$, let

$$br_{-j+1} = a_{-j}d + r_{-j}, \text{ where } 0 \leqslant r_{-j} < d.$$

This defines a sequence of digits a_{-j}, which automatically lie in the required range from 0 to $b-1$, and then $c/d = (a.\, a_{-1}a_{-2}\ldots)_b$.

This long division algorithm can be extended to complex bases to divide one Gaussian integer by another. However, the allowable remainders, r_{-j}, can be complex and they form a complicated set that depends on both the divisor and the base. That is, for each Gaussian integer divisor d and for each complex base b, there is some **remainder set** $\mathfrak{R}(d; b)$ of Gaussian integers such that the long division algorithm will yield a convergent radix expansion in base $b = -n + i$ with $0 \leqslant a_{-j} \leqslant n^2$ if and only if $r_{-j} \in \mathfrak{R}(d; b)$ for all $j \geqslant 0$.



(a) (b)

FIGURE 1. **(a) The remainder set for dividing Gaussian integers by 5 in base $-1+i$. (b) The remainder set for dividing by 3 in base $-1+i$.**

FIGURE 1 shows two examples of these remainder sets, namely $\mathcal{R}(5; -1+i)$ and $\mathcal{R}(3; -1+i)$. In the case of division by 3 in base $-1+i$ the remainder set $\mathcal{R}(3; -1+i)$ forms a complete residue system modulo 3; that is, this set tiles the plane by translations along Gaussian integers multiplied by 3. Therefore, at each stage of the long division algorithm, the remainder r_{-j} is uniquely determined and the resulting radix expansion will be unique.

In the case of division by 5 in base $-1+i$, the remainder set has 38 elements, which is more than the norm of the divisor. This means that in division by 5, there will sometimes be a choice for the remainder and so the algorithm will sometimes yield more than one radix expansion of the same number. This happens in the calculation of $(4+2i) \div 5$ in base $-1+i$. There is a choice of two remainders at the initial stage; either $4+2i = (1)5 + (-1+2i)$ or $4+2i = (1+i)5 + (-1-3i)$. After that, the algorithm is uniquely determined. The first alternative yields the following expansion.

$$\begin{aligned} 4+2i &= 1.5 + (-1+2i) \\ (-1+2i)(-1+i) = -1-3i &= 0.5 + (-1-3i) \\ (-1-3i)(-1+i) = 4+2i &= 1.5 + (-1+2i) \end{aligned}$$

The algorithm now repeats and so $(4+2i)/5 = (1.\overline{01})_{-1+i}$. The second alternative yields a different expansion.

$$\begin{aligned} 4+2i &= (1+i)5 + (-1-3i) \\ (-1-3i)(-1+i) = 4+2i &= 1.5 + (-1+2i) \\ (-1+2i)(-1+i) = -1-3i &= 0.5 + (-1-3i) \end{aligned}$$

The algorithm now repeats and, since $1+i = (1110)_{-1+i}$, it follows that $(4+2i)/5 = (1110.\overline{10})_{-1+i}$. The reader should try calculating the three expansions of $(-3-4i)/5$ in base $-1+i$. Each periodic expansion of period p can be evaluated by the standard method of multiplying by the pth power of the base and then subtracting the original expansion from it.

The above long division algorithm depends on first calculating the remainder sets $\mathcal{R}(d; b)$. Even though bounds can be put on their size, the exact determination of these remainder sets appears to be a tedious task. Are there other ways of doing division? The method of division given in [**1**] does not extend to complex bases. It essentially consists of finding inverses in the formal power series ring $\mathbb{Z}[[b^{-1}]]$ and then using the clearing algorithm. However, the inverse power series do not always converge when a complex base is substituted for b.

The complexity of the division can be appreciated by looking at the set of points which have the same initial expansion in a given complex base [**2**]. These subsets of the complex plane have fractal boundaries and it is not an easy task to determine whether a ratio of two Gaussian integers lies in a given set. Can the reader find either an easy method for calculating the remainder sets $\mathcal{R}(d; b)$ or find an alternative technique for doing division?

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Numerical Patterns and Geometrical Configurations

Warren Page
New York City Technical College
Brooklyn, NY 11201

Harold L. Dorwart
Trinity College
Hartford, CT 06106

> "... *there was a time when the study of configurations was considered the most important branch of all geometry.*"
> —David Hilbert

Today, in contrast to Hilbert's assertion, most students have limited knowledge about configurations—except perhaps Pappus' 9_3, Desargues' 10_3 and Petersen's $10_3 15_2$ (the logo on the cover of the Journal of Graph Theory). These configurations are shown in Figure 1. The configuration labeled 9_3 illustrates **Pappus' Theorem**: *If three points* $\{1,3,5\}$ *on one line are joined in consecutive order to three points* $\{4,6,2\}$ *on another line, the three intersection points* $\{9,7,8\}$ *are collinear.* The configuration 10_3 illustrates **Desargues' Theorem**: *When extended, the corresponding sides of triangles* $\langle 1,2,3\rangle$ *and* $\langle 4,5,6\rangle$ (*which are said to be in perspective from the point* 7), *meet in a set of collinear points* $\{8,10,9\}$ (*so that the triangles are said to be in perspective from a line also*).

In this note, we introduce and solve an interesting puzzle that serves to highlight the fascinating interplay between the geometric nature of point-and-line configurations and their representations as rectangular arrays of integers. We hope that our exposition will provide a rich source of challenges for students of varying levels of mathematical sophistication.

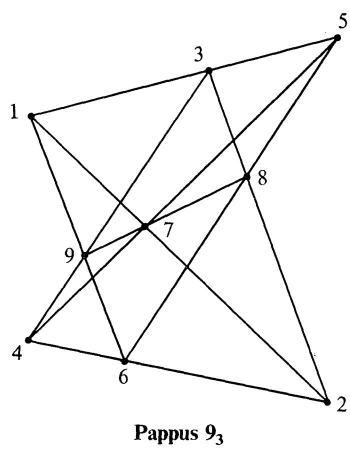
A number puzzle and a geometry puzzle

Using three sets of the first n integers $1,2,\ldots,n$, form a rectangular array of three rows by n columns such that:

($\mathbf{R}_1$) No integer is repeated within a column.

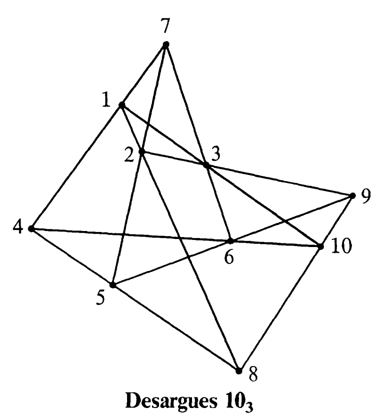
($\mathbf{R}_2$) No pair of integers occurs in two different columns.

A rectangular array which solves this **Number Puzzle** is called an $\boldsymbol{n_3}$**-table**. An n_3-table can exist



Pappus 9_3

Figure 1 (a)



Desargues 10_3

Figure 1 (b)

1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	1
4	5	6	7	8	9	10	11	1	2	3

FIGURE 2

1	2	3	4	5	6	7	8	9	10	11
3	1	11	7	2	4	5	10	6	9	8
8	10	9	1	6	11	3	4	7	5	2

FIGURE 3

only if $n > 6$. To be sure, suppose 1 belongs to columns $\begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$, $\begin{pmatrix} c \\ 1 \\ d \end{pmatrix}$, $\begin{pmatrix} e \\ f \\ 1 \end{pmatrix}$ of an n_3-table. Then, by $(\mathbf{R}_1)$ and $(\mathbf{R}_2)$, the integers $1, a, b, c, d, e, f$ must all be distinct.

Neither rule $(\mathbf{R}_1)$ nor $(\mathbf{R}_2)$ is violated when columns of an n_3-table are interchanged, or when integers within a column are interchanged. Tables obtained from one another by such interchanges are therefore considered to be the same. Thus, one may begin the search for an n_3-table by arranging the n integers in the first row in consecutive order; the second row can be taken as any arrangement of these n distinct integers which satisfies $(\mathbf{R}_1)$. Two 11_3-tables obtained in this manner are shown in FIGURES 2 and 3. (Although all of the integers $1, 2, \ldots, n$ can be made to appear in each row of an n_3-table for $n < 12$, this is not always possible for 12_3-tables ([**4**], 156).)

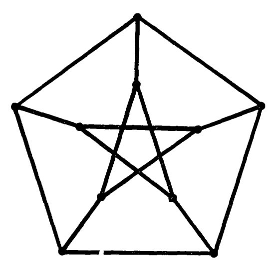
An n_3-table becomes another n_3-table under any permutation of the integers $1, 2, \ldots, n$. Two n_3-tables are said to be **equivalent** if one of them can be obtained from the other by some permutation of the integers. Nonequivalent n_3-tables will be referred to as being **distinct**. In 1887, Martinetti [**8**] showed how to generate $(n+1)_3$-tables from n_3-tables:

> If two columns C_1, C_2 of an n_3-table have no integer in common, replace one integer $c_1 \in C_1$ by $n+1$. Then replace an integer $c_2 \in C_2$ by $n+1$, where c_2 is chosen such that it and c_1 do not appear in any column of the n_3-table. An $(n+1)_3$-table will be formed by adjoining the new column of integers $\{c_1, c_2, n+1\}$ to the modified n_3-table.

Using a modification of this procedure and his knowledge of 8_3, 9_3, and 10_3-tables, Martinetti produced all thirty-one distinct 11_3-tables, two of which are shown in FIGURES 2 and 3. In 1895, von Sterneck [**10**] produced all two hundred and twenty-eight distinct 12_3-tables. Only isolated results of n_3-tables are known for $n > 12$. The form of the table in FIGURE 2 shows, however, that this type of n_3-table exists for all $n \geqslant 7$.

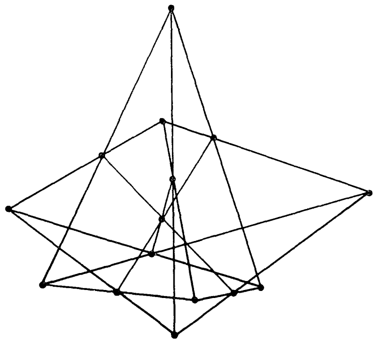
An ***n*₃-configuration** is a pattern in the Euclidean plane consisting of n points (i.e., vertices) and n lines such that three points lie on each line, and three lines pass through each point. The **Geometry Puzzle** is to draw n_3-configurations; the 9_3 and the 10_3 shown in FIGURE 1 are examples. According to Martinetti, as stated in [**1**], the Cremona-Richmond 15_3 (shown in FIGURE 4) is the simplest instance of an n_3-configuration that contains no triangles.

It will sometimes be important to distinguish between "n_3-configuration" (an unlabeled pattern as the 15_3-configuration) and "labeled n_3-configuration," where the pattern's points are labeled with the numbers $1, 2, \ldots, n$ (as in the 9_3 and 10_3-configurations in FIGURE 1). Two labeled n_3-configurations are said to be **equivalent** if one of them can be obtained from the other by some



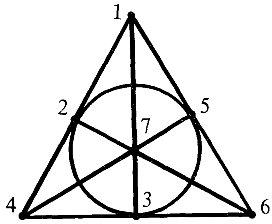
Peterson $10_3 15_2$

FIGURE 1 (c). **(Ten points, fifteen lines, three lines through each point, two points on each line.)**



Cremona-Richmond 15_3

FIGURE 4



Schematic figure for the 7_3 configuration:

1	**2**	**3**	**4**	**5**	**6**	**7**
2	**3**	**4**	**5**	**6**	**7**	**1**
4	**5**	**6**	**7**	**1**	**2**	**3**

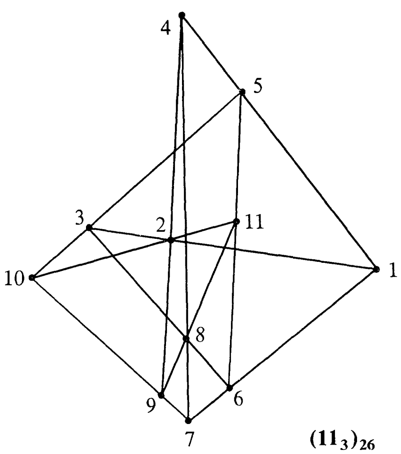
(Circle takes the place of the line 2 3 5-second column.)

FIGURE 5 (a)

permutation of the integers $1,2,\ldots,n$. Nonequivalent n_3-configurations are considered to be **distinct**.

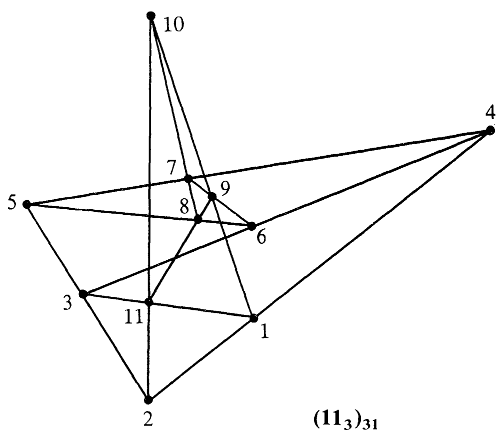
Each labeled n_3-configuration yields an n_3-table in a natural way: the numbers $1,2,\ldots,n$ in the n_3-table correspond to the labeled vertices of the n_3-configuration, and the n columns of the n_3-table correspond to the n lines of the n_3-configuration. Thus, the conditions in the definition of an n_3-configuration assure that conditions ($\mathbf{R}_1$) and ($\mathbf{R}_2$) can be satisfied. Not every n_3-table, however, can be realized in the Euclidean plane as an n_3-configuration. The unique 7_3-table, the unique 8_3-table, and one of the ten distinct 10_3-tables cannot be so realized. (Configurations corresponding to the three 9_3-tables are given in [**5**]; see [**2**] for the configurations corresponding to the nine realizable 10_3-tables). Using analytic methods, it can be shown ([**3**], pp. 125 and 147) that the equations of the lines are inconsistent for $n = 7$ and that they have solutions only in complex numbers when $n = 8$. Schematic figures for the 7_3 and 8_3-configurations are shown in FIGURE 5. Kantor [**6**] first showed that the 10_3-table designated as $(10_3)_4$ cannot be realized as a 10_3-configuration. The representation of $(10_3)_4$ in [**2**], however, is a reasonably good approximation (one line is not entirely straight) to a 10_3-configuration.

All thirty-one 11_3-tables can be realized as 11_3-configurations in the Euclidean plane. In 1894, von Sterneck [**9**] used standard techniques in projective geometry to produce all thirty-one 11_3-configurations. His drawings, reproduced from the original, are depicted on p. 88. The remainder of our discussion will center on 11_3's. Martinetti's 11_3-tables and von Sterneck's 11_3-configurations are related, as we shall see shortly.



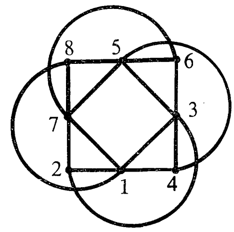
$(11_3)_{26}$

FIGURE 6



$(11_3)_{31}$

FIGURE 8



Schematic figure for the 8_3 configuration:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
2	**3**	**4**	**5**	**6**	**7**	**8**	**1**
4	**5**	**6**	**7**	**8**	**1**	**2**	**3**

(where, for example, line 2 3 5–second column–is represented by the portion of a circle 2 3 and the segment 3 5).

FIGURE 5 (b)

EXAMPLE. The labeled 11_3-configuration in FIGURE 6 (von Sterneck's twenty-sixth 11_3-configuration on page 88) describes the 11_3-table in FIGURE 7. And it will subsequently be shown how the 11_3-table in FIGURE 2 is known to be described by the labeled 11_3-configuration in FIGURE 8.

Suppose a given 11_3-table is known to be described by some yet undetermined labeling of a particular 11_3-configuration. It may still be quite difficult to label the 11_3-configuration so that its lines correspond with the columns of the 11_3-table. (How, for example, would one label $(11_3)_{26}$ if it was known *a priori* to be able to describe the 11_3-table in FIGURE 9?) This raises the somewhat fascinating question.

Can one match 11_3-configurations with 11_3-tables?

To solve this by trial-and-error would be almost impossible. In the next section, we introduce the notion of residue sets (apparently first used by Kantor [**6**]) as the crucial link for matching 11_3-tables with 11_3-configurations.

1	1	1	2	2	3	3	4	5	7	8
2	4	6	4	10	5	6	7	6	9	9
3	5	7	9	11	10	8	8	11	10	11

FIGURE 7

1	4	7	10	1	2	3	4	5	6	7
2	5	8	11	8	9	10	11	8	7	3
3	6	9	1	4	5	6	9	10	2	11

FIGURE 9

Residue sets

The integer 1 appears only in the first, fourth, and fifth columns of the 11_3-table in FIGURE 9. Accordingly, in an 11_3-configuration which describes this table, the point labeled 1 cannot lie on a line which contains a point labeled 5, 6, 7, 9 (the integers of the table which do not belong to any column containing 1). This set $R_1 = \{5, 6, 7, 9\}$ is called the **residue of 1**.

The **residue figure of 1**, denoted F_1, is the hypergraph whose vertices are the members of R_1, and whose edges are the lines (determined by the 11_3-table's columns) that join these vertices. FIGURE 10 shows the construction of residue figure F_1 for the 11_3-table in FIGURE 9. Similarly, for

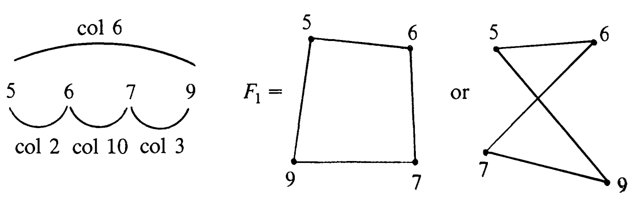


FIGURE 10

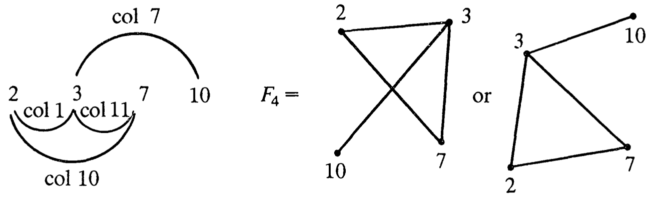


FIGURE 11

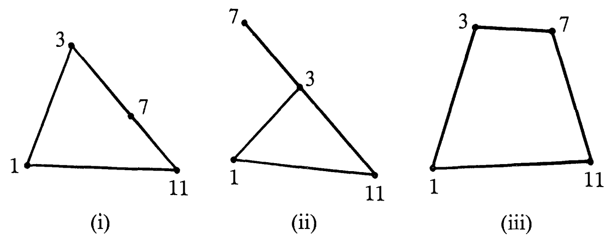


FIGURE 12

this 11_3-table, $R_4 = \{2, 3, 7, 10\}$ and FIGURE 11 shows the construction of F_4. Some care is needed in determining F_x, lest one overlook the possibility that three members of R_x may be collinear. In our example above, for instance, $R_5 = \{1, 3, 7, 11\}$ and the last column of the 11_3-table requires that F_5 be either FIGURE 12(i) or 12(ii), but not 12(iii).

Now let us establish that

there are six types of 11_3*-residue figures and all have exactly four vertices.*

This is based on the observation that for each integer x:

(A) R_x consists of four integers.

(B) Each member of R_x is connected to at least one other member of R_x.

(C) At most, only one member of R_x is connected to only a single member of R_x.

For verification of conditions (A)–(C), we may assume that x occurs in the first three columns

$$C_1 = \begin{pmatrix} x \\ a \\ b \end{pmatrix}, C_2 = \begin{pmatrix} c \\ x \\ d \end{pmatrix}, \text{and } C_3 = \begin{pmatrix} e \\ f \\ x \end{pmatrix}$$

of a given 11_3-table. Since each column contains three distinct members (by $(\mathbf{R}_1)$) and $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ are disjoint (by $(\mathbf{R}_2)$), it is clear that x is connected by lines of the 11_3-configuration to each of the vertices a, b, c, d, e, f. In particular, there are $(11-6)-1=4$ vertices not connected to x. To verify (B), assume to the contrary that g is not connected to any member of $R_x = \{g, h, i, j\}$. We may also assume that g belongs to columns C_4, C_5, C_6 of the 11_3-table, and, therefore, that h belongs to columns C_7, C_8, C_9 of the table. Then $(\mathbf{R}_1)$ and $(\mathbf{R}_2)$ imply that $i, j \in C_{10} \cap C_{11}$; but this contradicts rule $(\mathbf{R}_2)$. To verify (C), suppose $g \in R_x = \{g, h, i, j\}$ is connected only to h. Take $g \in C_4 \cap C_5 \cap C_6$ and $i \in C_7 \cap C_8 \cap C_9$. Since $j \notin C_4 \cup C_5 \cup C_6$, we must have $j \in C_7 \cup C_8 C_9$ as well as $j \in C_{10} \cap C_{11}$. Thus, i and j are connected, and h belongs to exactly one column ; each of the groupings $\{C_4, C_5, C_6\}$, $\{C_7, C_8, C_9\}$, $\{C_{10}, C_{11}\}$. In particular, i and j are also connected to h.

From (A) and (B), it follows that each F_x is a connected graph having 4, 5, or 6 edges (i.e., segments joining vertices). All such graphs are illustrated in FIGURE 13, as shown by the following argument. Assume that no three vertices $\{\alpha, \beta, \gamma, \delta\}$ of F_x are collinear. Then F_x has six edges if and only if it is of type a. Moreover, F_x has four or five edges if and only if it is obtained from type a by removing two or one edges subject to (B) and (C), and this requires that F_x be of type b, d, or e. Now consider the possibility that three vertices $\{\alpha, \beta, \gamma\}$ of F_x are collinear. Then, by (B)

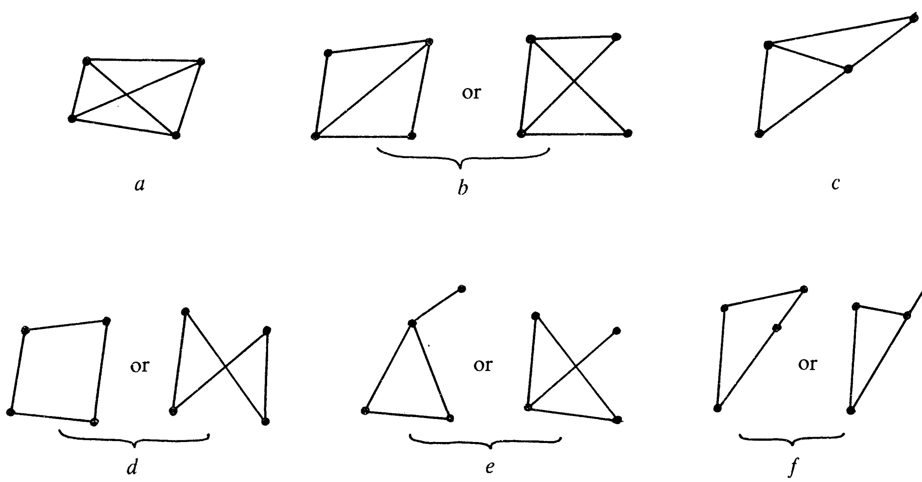


FIGURE 13

and (C), δ is connected with β (and therefore with α, or γ, or both α and γ), or else δ is connected with α and γ. Accordingly, F_x must be of type c or f.

The collection of residue figures $\{F_x\colon x=1,2,\dots,11\}$ is called the **residue set** of the 11_3-table (and of the 11_3-configuration(s) describing this table). By way of illustration, the residue set in FIGURE 7′ for the 11_3-table in FIGURE 7 consists of one type a, no b's, no c's, four d's, four e's, and two f's. Thus, it is designated as a (1,0,0,4,4,2)-residue set. One can also verify that the equivalent 11_3-table in FIGURE 9 has the (1,0,0,4,4,2)-residue set depicted in FIGURE 9′.

x	1	2	3	4	5	6	7	8	9	10	11
F_x	f	e	e	d	e	f	d	d	a	d	e

FIGURE 7′

x	1	2	3	4	5	6	7	8	9	10	11
F_x	d	d	a	e	f	d	d	e	f	e	e

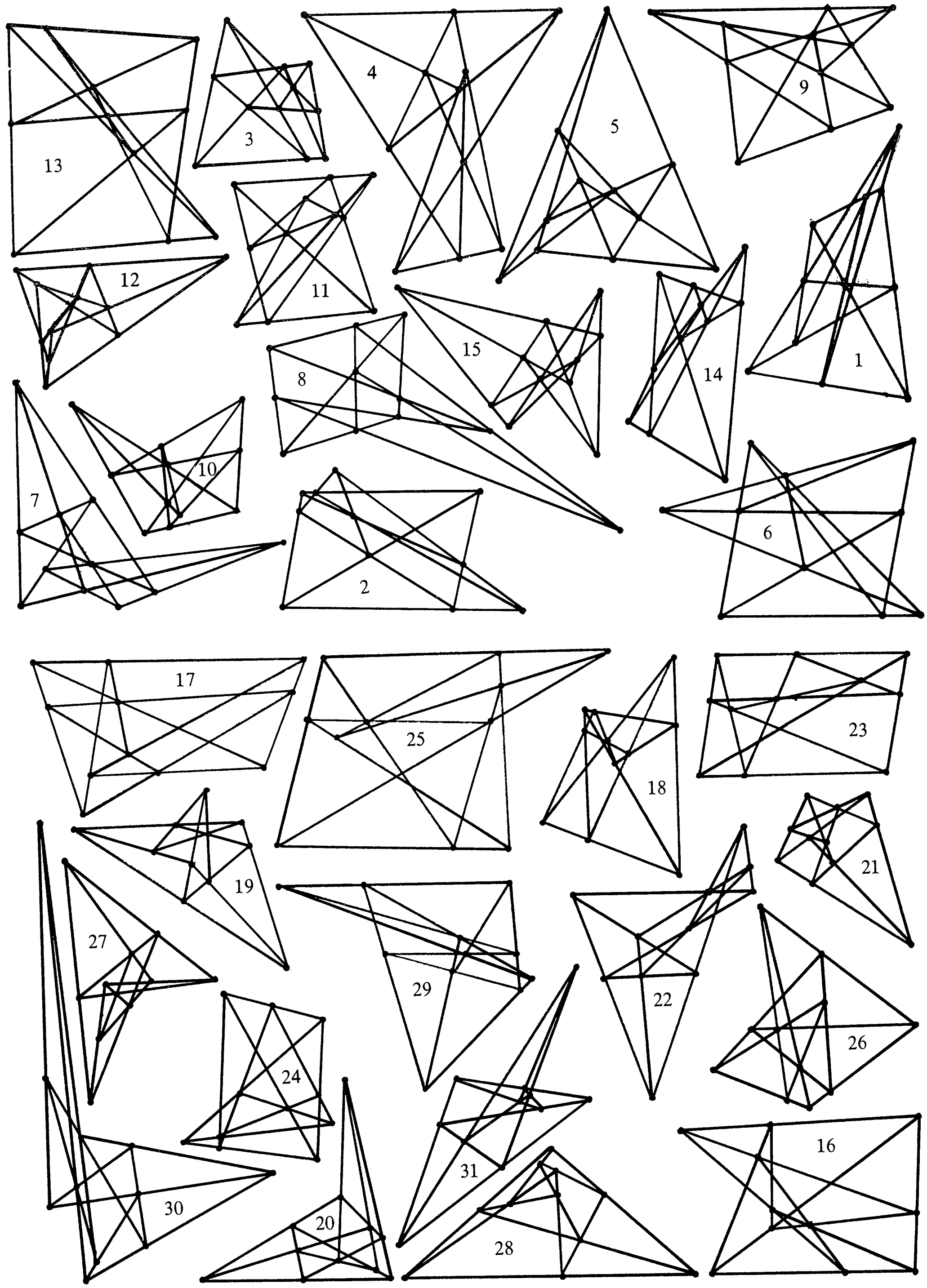
FIGURE 9′

Equivalent 11_3-tables (equivalent 11_3-configurations) have the same residue set since the structure of a hypergraph is unchanged when its vertices are relabeled. The complete list of residue sets for the thirty-one configurations shown on the next page (classified as 6-tuples) is given in TABLE 1.

Figure	a	b	c	d	e	f
1	0	3	0	0	4	4
2	0	4	0	3	4	0
3	0	5	2	1	2	1
4	2	2	2	1	4	0
5	0	3	2	2	4	0
6	0	3	0	2	2	4
7	0	3	4	0	2	2
8	0	2	0	0	4	5
9	1	0	0	3	2	5
10	0	1	0	1	6	3
11	0	2	0	1	6	2
12	0	2	2	1	2	4
13	0	2	0	0	4	5
14	0	2	0	3	4	2
15	0	2	4	0	2	3
16	0	2	0	5	2	2

Figure	a	b	c	d	e	f
17	0	2	0	3	4	2
18	0	4	0	1	0	6
19	0	0	8	0	0	3
20	0	3	1	3	2	2
21	1	0	1	1	2	6
22	2	0	4	1	0	4
23	0	2	0	1	6	2
24	0	1	0	4	6	0
25	0	2	0	3	4	2
26	1	0	0	4	4	2
27	0	2	4	1	4	0
28	1	0	1	0	0	9
29	1	0	1	3	6	0
30	0	6	2	3	0	0
31	0	0	11	0	0	0

TABLE 1. **The residue sets of 11_3-configurations.**



The thirty-one 11_3-configurations from Tables I and II of reference [9].

Interlacing the basic puzzles

We begin by illustrating how to relabel configuration $(11_3)_{26}$ in FIGURE 6 so that its lines correspond to the columns of the 11_3-table in FIGURE 9. For convenience, integers in the 11_3-table will be referred to as "numbers," and points of the 11_3-configuration will be viewed as "positions." Position x will be denoted $\langle x \rangle$ when serving as storage for numbers from FIGURE 9.

We know from FIGURE 7′ that position 9 has an a-type residue figure. And since (FIGURE 9′) number 3 also has an a-type residue figure, we place number 3 into position 9. Thus, $3 \in \langle 9 \rangle$ in FIGURE 6 (meaning that we replace label 9 in FIGURE 6 with a 3). Similarly, positions 1 and 6 have f-type residue figures, and numbers 5 and 9 have f-type residue figures. Therefore $\{5,9\} \in \langle 1 \rangle \cup \langle 6 \rangle$ in some order in FIGURE 6. Before attempting to decide the order, note (FIGURE 9) that numbers 2, 5, 9 must be collinear. Therefore $2 \in \langle 7 \rangle$ and (since numbers 1, 2, 3 must also be collinear) $1 \in \langle 10 \rangle$. Numbers 1, 2, 6, 7 should be placed somewhere in positions $\langle 4 \rangle$, $\langle 7 \rangle$, $\langle 8 \rangle$, $\langle 10 \rangle$ since these numbers and positions each have d-type residue figures. Since $1 \in \langle 10 \rangle$ and $2 \in \langle 7 \rangle$, this leaves $\{6,7\} \in \langle 4 \rangle \cup \langle 8 \rangle$. For e-type residue figures, we have $\{4, 8, 10, 11\} \in \langle 2 \rangle \cup \langle 3 \rangle \cup \langle 5 \rangle \cup \langle 11 \rangle$. To complete the picture, we now consider those columns (FIGURE 9) that contain each number still remaining to be placed in positions in FIGURE 6. Based on what is already known (and indicated in FIGURE 6), one can determine which positions are available for each number. For example, from columns 2, 5, and 8 of the table in FIGURE 9,

$$C_2 \Rightarrow 4 \in \langle 3 \rangle \cup \langle 5 \rangle, C_5 \Rightarrow 4 \in \langle 2 \rangle \cup \langle 3 \rangle \cup \langle 5 \rangle \cup \langle 11 \rangle, C_8 \Rightarrow 4 \in \langle 3 \rangle \cup \langle 5 \rangle$$

requires that $4 \in \langle 3 \rangle \cup \langle 5 \rangle$. After noting this in FIGURE 6, one proceeds to

$$C_3 \Rightarrow 8 \in \langle 3 \rangle \cup \langle 5 \rangle, C_5 \Rightarrow 8 \in \langle 3 \rangle \cup \langle 5 \rangle, C_9 \Rightarrow 8 \in \langle 2 \rangle \cup \langle 3 \rangle \cup \langle 5 \rangle \cup \langle 11 \rangle,$$

and thus obtains $8 \in \langle 3 \rangle \cup \langle 5 \rangle$. Then, from

$$C_4 \Rightarrow 10 \in \langle 3 \rangle \cup \langle 5 \rangle \cup \langle 2 \rangle \cup \langle 11 \rangle, C_7 \Rightarrow 10 \in \langle 2 \rangle \cup \langle 11 \rangle, C_8 \Rightarrow 10 \in \langle 8 \rangle \cup \langle 2 \rangle \cup \langle 4 \rangle \cup \langle 11 \rangle,$$

we obtain $10 \in \langle 2 \rangle \cup \langle 11 \rangle$. Also, $11 \in \langle 2 \rangle \cup \langle 11 \rangle$ follows from this and the observation that

$$C_4 \Rightarrow 11 \in \langle 2 \rangle \cup \langle 11 \rangle, C_8 \Rightarrow 11 \in \langle 8 \rangle \cup \langle 2 \rangle \cup \langle 11 \rangle, C_{11} \Rightarrow 11 \in \langle 2 \rangle \cup \langle 11 \rangle.$$

Putting this all together, we see that the 11_3-table in FIGURE 9 is described by both of the labeled $(11_3)_{26}$-configurations in FIGURE 14. These configurations are considered "identical" since (by analogy with 11_3-tables) three points lie on a line in FIGURE 14(i) if and only if they lie on a line in FIGURE 14(ii).

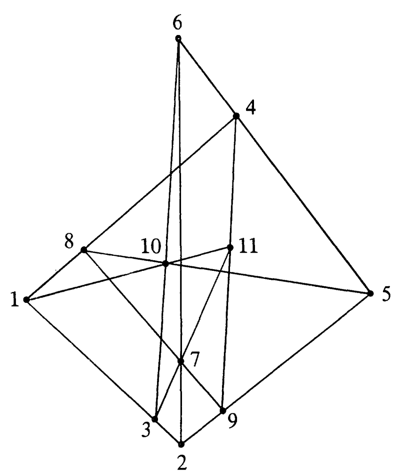


FIGURE 14 (i)

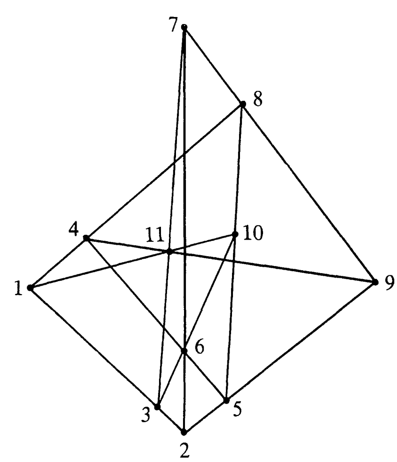


FIGURE 14 (ii)

If a labeled 11_3-configuration $\mathscr{C}$ describes an 11_3-table T, then any permutation of the integers in $\mathscr{C}$ produces the same relabeling of the integers in T, and conversely. Based on this, we can almost answer the question:

How does one determine a relabeling (i.e., permutation) between equivalent 11_3-tables?

Suppose T_1 and T_2 are equivalent 11_3-tables (as in FIGURES 7 and 9) and suppose $\mathscr{C}_1$ is a labeled 11_3-configuration which describes T_1 (as in FIGURE 6). Then, by the preceding technique, we can obtain a labeled 11_3-configuration $\mathscr{C}_2$ that describes T_2 (say, FIGURE (14(i), for example). The 1–1 mapping of vertices π: $\mathscr{C}_1 \to \mathscr{C}_2$ transforms T_1 into T_2. If $\mathscr{C}_3$ is another labeled 11_3-configuration that describes T_2 (say, the configuration in 14(ii)), then the permutation ν: $\mathscr{C}_1 \to \mathscr{C}_3$ also transforms T_1 into T_2. In our example, it is easy to verify that each of the permutations

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 5 & 10 & 8 & 6 & 4 & 9 & 2 & 7 & 3 & 1 & 11 \end{pmatrix}$$

$$\nu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 11 & 4 & 7 & 8 & 5 & 2 & 6 & 3 & 1 & 10 \end{pmatrix}$$

transforms the 11_3-table in FIGURE 7 into the 11_3-table in FIGURE 9.

Now consider the case where it is known that equivalent 11_3-tables T_1 and T_2 can be described by labelings of a given configuration $(11_3)_m$ on page 88. If $\mathscr{C}_0$ is an arbitrarily labeled $(11_3)_m$ and T_0 is the 11_3-table described by $\mathscr{C}_0$, we can proceed as above to produce a relabeling $\mathscr{C}_1$ of $\mathscr{C}_0$ such that $\mathscr{C}_1$ describes T_1. Repeating this process with $\mathscr{C}_1$, T_1 and T_2, we obtain the desired permutation which maps T_1 to T_2.

To completely solve the problem raised above, we must determine exactly which one of the thirty-one distinct 11_3-configurations will have its labelings describe a given 11_3-table. Once this is done, we have completely solved the puzzle:

Which 11_3-configurations describe, and are described by, which 11_3-tables?

Let R be the collection of 6-tuples (see TABLE 1), each of which has exactly one 11_3-configuration associated with it. Then each residue 6-tuple in R has exactly one (equivalence) class of equivalent 11_3-tables associated with it. If the residue set of an 11_3-table belongs to R, the 11_3-configuration associated with this residue set is the candidate sought. Some work may be required when an 11_3-table's residue set does not belong to R. We illustrate, for example, how to choose between $(11_3)_8$ and $(11_3)_{13}$ for the 11_3-table with (0,2,0,0,4,5)-residue set shown in FIGURE 15.

1	4	7	10	2	8	3	8	5	11	6
2	5	8	11	5	10	4	11	9	3	9
3	6	9	1	7	2	7	4	10	6	1

x	1	2	3	4	5	6	7	8	9	10	11
F_x	b	e	f	b	e	f	f	e	e	f	f

FIGURE 15. **An 11_3-table and its corresponding (0,2,0,0,4,5)-residue set.**

Begin by arbitrarily assigning integers $1, 2, \ldots, 11$ to the points of each configuration $(11_3)_8$ and $(11_3)_{13}$ as in FIGURE 16. Then write the corresponding 11_3-tables and find their residue sets.

If we attempted to match the 11_3-table in FIGURE 15 with $(11_3)_8$, then (by considering b-type residue figures) numbers 1 and 4 in the residue set of FIGURE 15 must go into positions $\langle 4 \rangle$ and $\langle 8 \rangle$ in $(11_3)_8$. But this is impossible since it requires that 1 and 4 lie on a line in $(11_3)_8$ while not appearing together in any single column of the table in FIGURE 15. Thus, the 11_3-table in FIGURE 15 must be determined by a labeling of $(11_3)_{13}$. And now it is a simple matter to corroborate that it is the labeling shown in FIGURE 17.

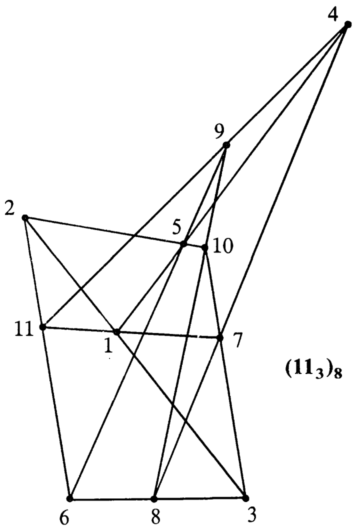


FIGURE 16 (a)

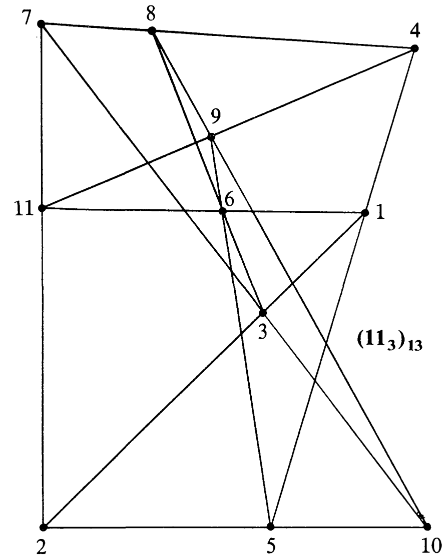


FIGURE 16 (b)

1	1	2	3	3	4	5	8	2	4	1
2	4	5	6	7	7	6	9	6	9	7
3	5	10	8	10	8	9	10	11	11	11

x	1	2	3	4	5	6	7	8	9	10	11
F_x	f	f	f	b	e	e	f	b	f	e	e

1	1	2	3	3	4	5	8	2	4	1
2	4	5	6	7	7	6	9	7	9	6
3	5	10	8	10	8	9	10	11	11	11

x	1	2	3	4	5	6	7	8	9	10	11
F_x	f	b	f	e	e	e	f	b	f	f	e

FIGURE 16. **Assignment of labels to configurations $(11_3)_8$ and $(11_3)_{13}$, the 11_3-tables determined by the labelings, and their corresponding residue sets.**

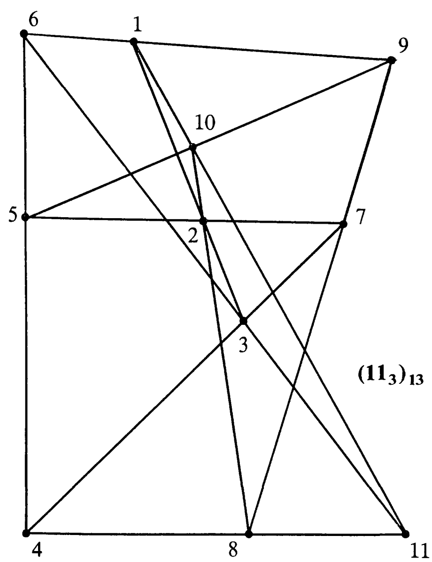


FIGURE 17

Readers can proceed similarly for the (0,2,0,1,6,2)-residue set associated with $(11_3)_{11}$ and $(11_3)_{23}$, and the (0,2,0,3,4,2)-residue set associated with $(11_3)_{14}$, $(11_3)_{17}$, and $(11_3)_{25}$.

Although considerable attention has been given to n_3-configurations for the cases where $n = 7,8,9,10$ (see **[1]**, **[5]**, **[7]**), there appears to be no reference in the recent literature to work done on the 11_3's about a century ago. Thus, our puzzle's focus on the case of $n = 11$ has the added benefit of summarizing and updating some important earlier results on the 11_3's.

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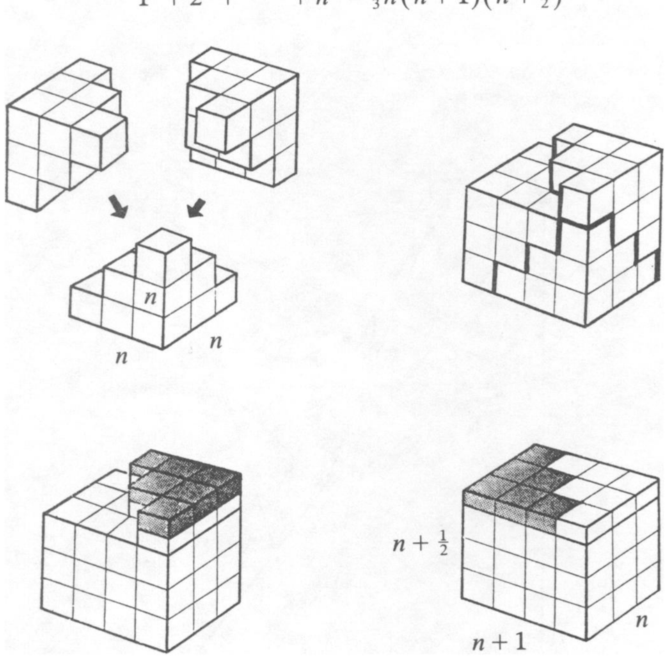
Proof without words: Sum of squares

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{3}n(n+1)\left(n+\tfrac{1}{2}\right)$$



—Man-Keung Siu
University of Hong Kong

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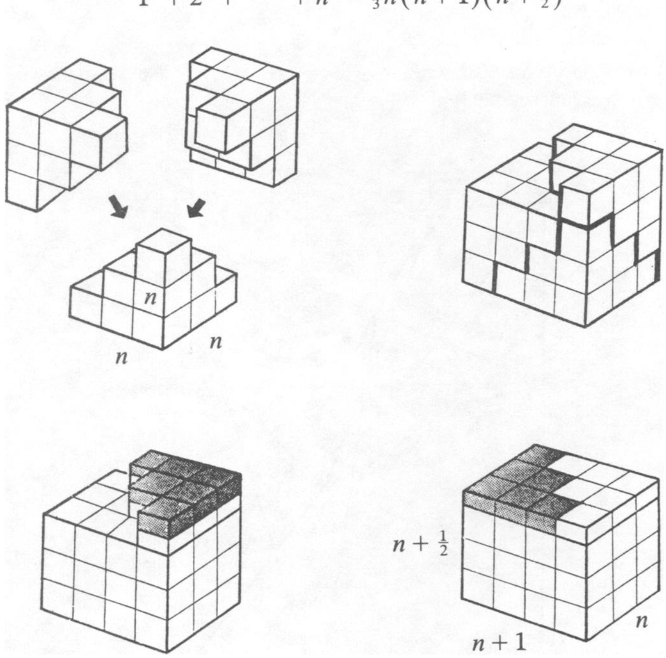
$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{3}n(n+1)(n+\tfrac{1}{2})$$



—Man-Keung Siu
University of Hong Kong

The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof

THOMAS YUSTER
Middlebury College
Middlebury, VT 05753

One of the most simple and successful techniques for solving systems of linear equations is to reduce the coefficient matrix of the system to **reduced row echelon form**. This is accomplished by applying a sequence of elementary row operations (see, e.g., [**1**, p. 5]) to the coefficient matrix until a matrix B is obtained which satisfies the following description:

> If a row of B does not consist entirely of zeros then the first nonzero number in the row is a 1 (usually called a leading 1).
>
> If there are any rows that consist entirely of zeros, they are grouped together at the bottom of B.
>
> In any two successive non-zero rows of B, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
>
> Each column of B that contains a leading 1 has zeros everywhere else.

The matrix B is said to be in reduced row echelon form.

It is well known that if A is an $m \times n$ matrix and x is an $n \times 1$ vector, then the systems $Ax = 0$ and $Bx = 0$ have the same solution set. However, the solution to the system $Bx = 0$ may be read off immediately from the matrix B. For example, consider the system:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 2x_1 + 3x_2 + 7x_3 &= 0 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \end{aligned}$$

The matrix of coefficients is

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{pmatrix}$$

and through use of elementary row operations we obtain

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is seen to be in reduced row echelon form. Column 3 (unlike columns 1 and 2) does not contain a leading 1. We call such a column a **free** column, because in the solution to the system $Bx = 0$ (and hence $Ax = 0$), the variable corresponding to that column is a free parameter. Thus we can set $x_3 = t$, and read off the equations $x_1 + 2t = 0$ (row 1) and $x_2 + t = 0$ (row 2). The solution set is then $x_1 = -2t, x_2 = -t, x_3 = t$.

An important theoretical result is that the reduced row echelon form of a matrix is unique. Most texts either omit this result entirely or give a proof which is long and very technical (see [**2**, p. 56]). The following proof is somewhat clearer and less complicated than the standard proofs.

THEOREM. *The reduced row echelon form of a matrix is unique.*

Proof. Let A be an $m \times n$ matrix. We will proceed by induction on n. For $n = 1$ the proof is obvious. Now suppose that $n > 1$. Let A' be the matrix obtained from A by deleting the nth column. We observe that any sequence of elementary row operations which places A in reduced row echelon form also places A' in reduced row echelon form. Thus by induction, if B and C are reduced row echelon forms of A, they can differ in the nth column only. Assume $B \neq C$. Then there is an integer j such that the jth row of B is not equal to the jth row of C. Let u be any column vector such that $Bu = 0$. Then $Cu = 0$ and hence $(B - C)u = 0$. We observe that the first $n-1$ columns of $B - C$ are zero columns. Thus the jth coordinate of $(B - C)u$ is $(b_{jn} - c_{jn})u_n$. Since $b_{jn} \neq c_{jn}$ we must have $u_n = 0$. Thus any solution to $Bx = 0$ or $Cx = 0$ must have $x_n = 0$. It follows that both the nth columns of B and C must contain leading 1's, for otherwise those columns would be free columns and we could arbitrarily choose the value of x_n. But since the first $n-1$ columns of B and C are identical, the row in which this leading 1 must appear must be the same for both B and C, namely the row which is the first zero row of the reduced row echelon form of A'. Because the remaining entries in the nth columns of B and C must all be zero, we have $B = C$, which is a contradiction. This establishes the theorem.

We remark that this proof easily generalizes to the following proposition: *Let A be an $m \times n$ matrix with row space W. Then there is a unique $m \times n$ matrix B in reduced row echelon form such that the row space of B is W.* (For another proof, see [**2**, p. 56].)

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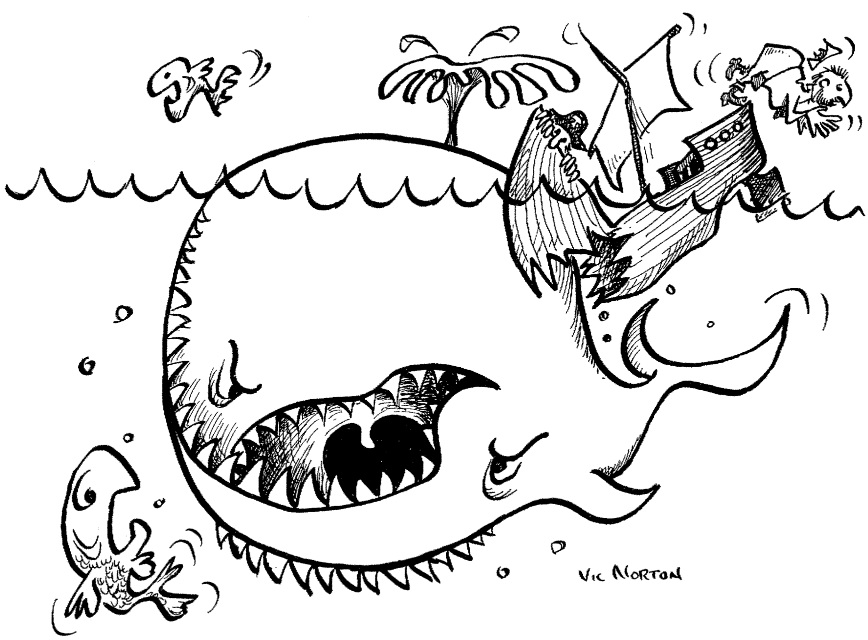

Riddle: What is non-orientable and lives in the sea?

—ROBERT MESSER

(See News and Letters if you give up.—ed.)

An Application of Graph Theory and Integer Programming: Chessboard Non-attacking Puzzles

L. R. Foulds
University of Canterbury
New Zealand

D. G. Johnston
Massey University
New Zealand

How many queens can be placed on a chessboard and where are they to be placed if no two are allowed to attack each other? This problem and similar puzzles for the other chess pieces have intrigued mathematicians for at least 130 years. Although the puzzles are interesting in themselves, the main purpose of this note is to use them to introduce two topics from discrete mathematics not yet part of the usual undergraduate mathematics curriculum but of growing importance in industrial mathematics: maximal cliques from graph theory and 0-1 integer programming.

Consider a regular chessboard and the five chess pieces: rook (castle), bishop, knight, king, and queen. In the game of chess each player begins with two of each of the first three of those pieces and one each of the last two. A large collection of puzzles can be posed on the chessboard in which a person is assumed to have a large number of pieces of one type and none of the others. One problem is to decide how to place the maximum number of pieces of the same kind on the board (no more than one to a square) so that no capturing can occur if the pieces are played according to the usual legal moves.

As an example, let's consider the rook puzzle. We wish to place as many rooks as possible on the chessboard so that no two attack each other. A rook moves orthogonally, that is, through any number of unoccupied squares parallel to an edge of the board. Thus once a rook is placed it will not be possible to place another on the same rank (row of horizontal squares) or file (column of vertical squares). Thus the problem reduces to finding a set of squares such that:

(i) No two are in the same rank or file.
(ii) The number in the set is a maximum among all sets satisfying property (i).

Since a chessboard has only eight ranks (and files) it is obvious that no more than eight rooks can be placed. Hence the eight squares comprising the leading diagonal (see Figure 3(a)) constitute a set satisfying properties (i) and (ii). This is only one of many optimal solutions to the problem—the others can be arrived at by interchanging the ranks of various pairs of rooks in the leading diagonal solution.

This puzzle of the rooks is very easy to solve. We shall see that the puzzles associated with the other pieces are more challenging. Combinatorial methods (e.g., Yaglom and Yaglom [**25**]) can be used to discover how many solutions there are to a given problem. In the discussion that follows we present methods of solutions based on graph theory and operations research which construct actual solutions. There is no suggestion that these methods are superior to other methods for solving the chessboard problems. Rather, the problems are used as a vehicle to introduce the reader to interesting and important topics in graph theory and integer programming.

It is appropriate to begin with a brief history of these puzzles and their solutions. The classic queens' puzzle (our opening question) originated with Gauss in 1850, where eight was quickly found to be the answer. But for the next sixty years, mathematicians were mainly concerned with the "Eight Queens Problem," that is, the problem of enumerating how many different solutions to the queens' puzzle exist. At first Gauss concluded there were 76, then changed his mind to 72 and finally arrived at 92, which is correct. In 1910 G. Bennett [**2**] concluded that there are only 12 distinctly different solutions to the queens' problem (i.e., solutions that could not be obtained one from another by rotation or reflection of the chessboard); this was later proved by T. Gosset in 1914 [**10**].

The "Eight Queens Problem" was eventually extended to "The n-Queens Problem," that is, solving the queens' puzzle for the general $n \times n$ chessboard. This was mentioned as early as 1874 by J. W. Glaisher [**8**] who attempted to solve it using determinants. The answer was always suspected to be n queens, but a clear proof was not provided until 1969 by Hoffman, Loessi and Moore [**12**]. Other variations on this problem have been developed, including an application to an $n \times n$ chessboard which has had its opposite sides "identified," creating the surface of a torus. (See [**16**].)

Many other problems using the chess pieces have been devised. For example, to solve the "knight's tour" problem, a knight must tour the board using only legal moves, landing on every square exactly once. Another problem is to find the minimum number of pieces of one particular kind such that every square is either occupied or attacked by at least one piece; and in addition count the number of different solutions. Kraitchik has done a lot of work on the first problem in his book, *Mathematical Recreations* [**15**] and recently R. W. Robinson produced a paper [**17**] concerning the second problem. Some of the more important publications are: Berge [**3**], Berman [**4**], Hansche and Vucenic [**11**], Huff [**13**], Klarner [**14**], Sebastian [**19**], Selfridge [**20**], Shen and Shen [**21**], and Slater [**22**].

Using maximal cliques to solve the puzzles

In this section we apply graph theory to analyze the non-attacking puzzles introduced in the previous section. We first review the ideas necessary to solve the puzzles in a graph-theoretic way. A **graph** G comprises a finite nonempty set, V, of **vertices** and a set, E, of unordered pairs of distinct points of V, called **edges**. If $u, v \in V$ and $\{u, v\} \in E$, then vertices u and v are said to be **adjacent** and each is said to be **incident** with the edge $\{u, v\}$. Although a graph is often represented by a diagram of dots (corresponding to vertices in V) with arcs connecting some dots (edges joining incident vertices), the graph itself is independent of the spatial arrangement of the dots and the actual layout of the arcs. That is, a graph is uniquely specified by specifying its vertices and which pairs of vertices are adjacent.

Given any graph G with n vertices, it is possible to create a new graph $\bar{G}$, with the same n vertices called the **complement** of G. This can be done as follows. Construct initially a graph with n vertices and every pair of distinct vertices adjacent (there are $n(n-1)/2$ edges). This graph, denoted K_n, is called the **complete graph** on n vertices. Next, delete from K_n all edges which appear in G; the graph that remains is $\bar{G}$. Thus $\bar{G}$ has all the edges that do not appear in G and any pair of vertices of G will be adjacent in exactly one of G and $\bar{G}$. Diagrams of some graphs and their complements are shown in FIGURE 1.

A standard application of graphs is to model associations between a given set of people. Suppose each person at a party is represented by a vertex. We create a graph by specifying that an edge is incident with vertices u and v if and only if the people represented by u and v know each other. Unless all at the party are strangers to each other, there is at least one subset of the people present with the property that everyone in the subset knows everyone else in the subset. Such a group is commonly called a **clique**, and in the graph model is represented by a subset of the

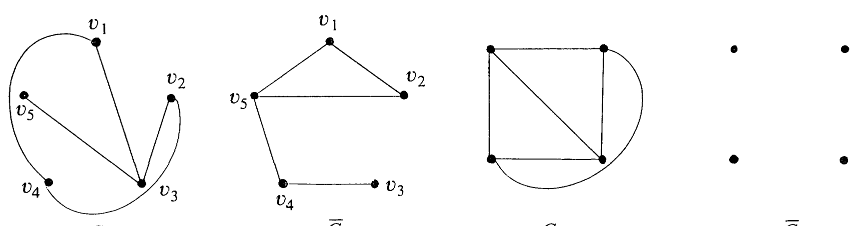

FIGURE 1. **Two graphs and their complements.**

vertices (also called a clique) such that every vertex in the subset is adjacent to every other vertex in the subset. A clique with the property that it no longer remains a clique when any further vertex is added is called a **maximal clique**. It is interesting to note that not all maximal cliques in a given graph have the same number of vertices. The graphs at the left in FIGURE 1 illustrate this, where the sets $\{v_1, v_2, v_5\}$ and $\{v_3, v_4\}$ constitute maximal cliques in $\bar{G}$.

We can now use this graph-theoretic machinery to analyze the chessboard puzzles. The approach is the same for all chess pieces. We begin by taking an arbitrary piece P, and creating a graph G with 64 vertices—one for each square on the board. We then examine each square s in turn and find which other squares would be attacked if P was placed on s. Edges are drawn in G according to the following rule: If a piece P on square s can attack a piece P on square t, then an edge joins s and t. When this has been done for all 64 squares we have created a graph for the puzzle concerned with the given piece P. The puzzle now reduces to finding a set A of vertices in G with the properties:

(i) No two vertices in A are adjacent.
(ii) The number of vertices in A is a maximum.

It will become evident that for any choice of piece P, there are many sets which satisfy (i) and (ii). There are many solutions to each puzzle.

Suppose we have found a set A satisfying (i) and (ii). Then (i) implies that each pair of vertices in A must be adjacent in $\bar{G}$, i.e., A must constitute a clique in $\bar{G}$. But (ii) implies that A must be a clique that has the maximum number of vertices among all cliques in $\bar{G}$. Therefore the puzzle becomes one of finding the maximal clique with the most vertices in $\bar{G}$.

The reader may wonder why we have gone to so much trouble to transform the puzzle into another one which looks just as difficult. The point is that there are standard methods in graph theory for finding maximal cliques in any given graph. Thus we solve the chessboard puzzle by using a method to generate all the maximal cliques of $\bar{G}$ and then single out those having the maximum number of vertices. Such a maximal clique generating method has been presented by Bron and Kerbosch [5] and we utilized their technique as a subroutine in a computer program written to solve the puzzles. The program generated the actual graphs G and $\bar{G}$. These graphs for a 3×3 board are shown in FIGURE 2, together with the corresponding puzzle solution they produce.

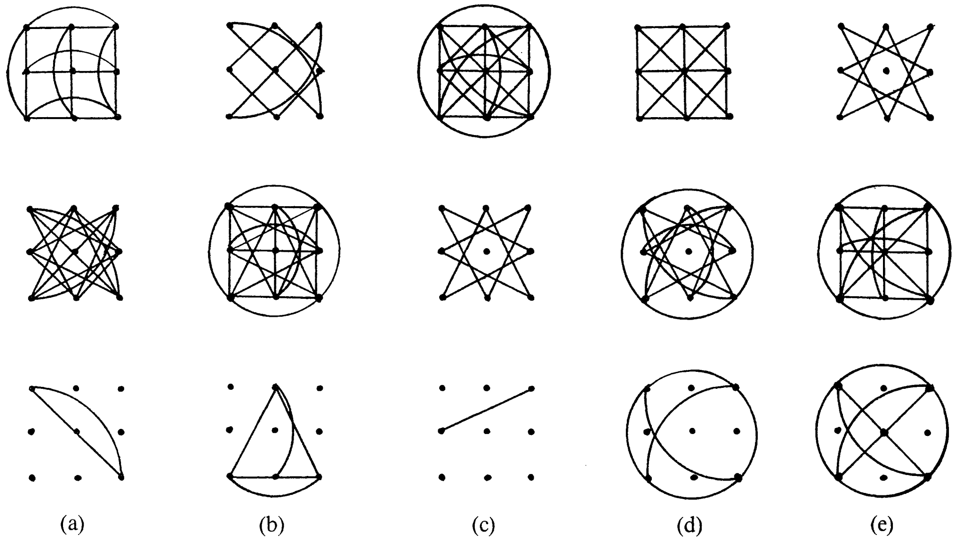

FIGURE 2. **The 3×3 chessboard puzzles: top row, graphs; middle row, their complements; bottom row, maximal cliques and corresponding solutions. (a) rooks, (b) bishops, (c) queens, (d) kings, and (e) knights.**

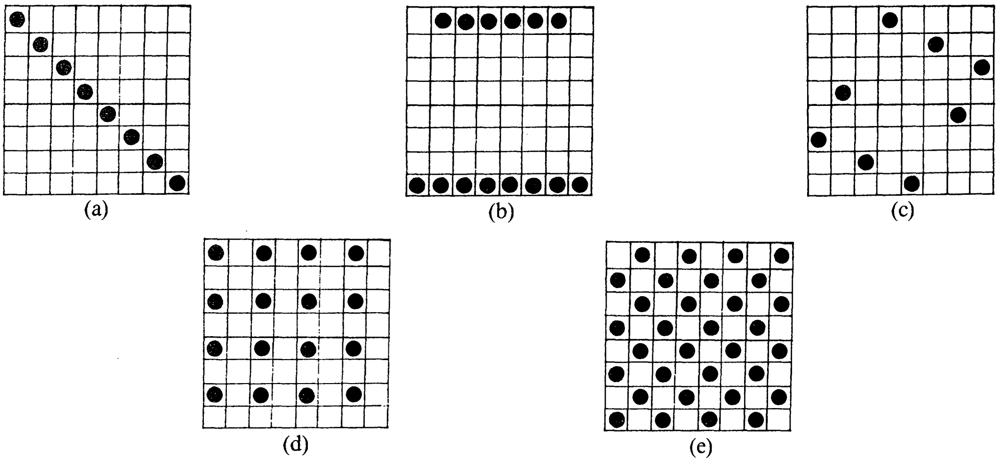

FIGURE 3. **Sample solutions for the puzzles on an 8 × 8 board for (a) rooks, (b) bishops, (c) queens, (d) kings, and (e) knights.**

The method of Bron and Kerbosch is a backtracking approach, i.e., in this case it explores various combinations of edges and cuts off lines of search which it establishes cannot lead to a clique. It generates cliques in a rather unpredictable order so as to try to minimize the number of search paths. It seems to produce the large cliques first and it generates sequentially the cliques having a large common intersection.

Sample solutions to the puzzles for 8 × 8 boards are displayed in FIGURE 3.

Using integer programming to solve the puzzles

We turn now to a completely different way of solving the puzzles. **Linear programming** is a well-known technique of operations research for maximizing or minimizing the value of a linear function of a number of nonnegative variables subject to a system of linear constraints. When the variables are further constrained to be integers, the problem becomes what is known as an **integer programming** problem. Thus the general maximizing integer programming problem with m constraints and n variables can be stated as:

$$\text{Maximize} \quad c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{1}$$

$$\text{subject to the conditions:} \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}s_n &= b_2, \\ \vdots \qquad\qquad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{2}$$

$$x_1, x_2, \ldots, x_n \text{ must be nonnegative,} \tag{3}$$

$$x_1, x_2, \ldots, x_n \text{ must be integers.} \tag{4}$$

Without loss of generality we may *assume the b_j's and a_{ij}'s to be given integers* rather than rational numbers.

Sometimes a further restriction is placed on the variables:

$$x_i = 0 \text{ or } 1, \quad i = 1, 2, \ldots, n. \tag{5}$$

Problems with this further restriction are called **zero-one (0-1) integer programming** problems.

Since the number of possible values for each variable is extremely limited, the reader may suspect that 0-1 integer programming problems are easier to solve than linear programming problems. Unfortunately this is not so. While the efficient simplex method (see [**23**]) exists for solving linear programming problems, no such efficiency is available for solving integer programming problems. The development of improved (more efficient) algorithms for solving integer programming problems is an active area in operations research. (See Balas [**1**], Dakin [**6**], and Gomory [**9**] for basic methods.) Sometimes, however, the efficiency of the simplex method can be salvaged as follows. The integer programming problem given by the conditions (1)–(4) is very similar in structure to the linear programming problem given by (1)–(3). Thus we can see what happens if we solve an integer programming problem by temporarily ignoring condition (4) and solving the less restricted linear programming problem. This approach can also be used for the 0-1 integer programming problem by ignoring (5). If the optimal solution to the corresponding linear programming problem obeys (4) or (5), then it is optimal for the integer programming problem or the 0-1 integer programming problem respectively. We now derive conditions for when this will occur based on section 3.3 of Garfinkel and Nemhauser [7].

We begin by expressing the problem in matrix form. Let $X=(x_1, x_2,\dots, x_n)^T$, $A=(a_{ij})$, the coefficient matrix of the system (2), and $b=(b_1, b_2,\dots, b_m)^T$, with all b_j and a_{ij} integers. Then (2) and (3) can be expressed as:

$$AX=b, \quad X\geqslant 0. \tag{6}$$

A maximal set of linearly independent columns of the $m\times n$ matrix A is called a **basis** (for the column space) of A. In order to derive a criterion for integer solutions to the system (2), we assume that the matrix A has a basis of m columns (i.e., we assume rank $A=m$). (This will be true when the m rows of A are linearly independent, and when they are dependent, a maximal linearly independent set of equations can be chosen from (2) to produce a coefficient matrix with fewer rows, corresponding to an equivalent system of equations.)

Choose m columns of A which form a basis; these form a nonsingular submatrix B of A, called a **basis submatrix** of A. By re-ordering the variables x_i if necessary, we can assume that the first m columns of A form the submatrix B, and the remaining columns form a submatrix N. Then (6) can be written as

$$BX_B+NX_N=b. \tag{6$'$}$$

A **basic solution** of (6) is defined as

$$\begin{aligned} X_B&=B^{-1}b,\\ X_N&=0. \end{aligned} \tag{7}$$

Hence if B^{-1} is an integer matrix, the corresponding basic solution of (6) is integer. We shall now show that B^{-1} will be an integer matrix whenever A is what is termed totally unimodular. A square, integer matrix C is **unimodular** if its determinant is 1 or -1. An integer matrix A is **totally unimodular** if *every* square, nonsingular submatrix of A (gotten by striking out any $n-k$ columns and any $m-k$ rows of A, $1\leqslant k<$ rank A) is unimodular. The following theorem gives what we need.

THEOREM 1. *If the matrix A is totally unimodular, then every basic solution of* (6) *is integer.*

Proof. If B is any basis submatrix of A, then B is nonsingular, so B^{-1} can be expressed as

$$B^{-1}=(\det B)^{-1}\bar{B}$$

where $\bar{B}$ is an integer matrix. Since A is totally unimodular, B must be unimodular, hence B^{-1} is an integer matrix. Since the entries in the vector b are integer, (7) shows that every basic solution is integer.

We note that while our earlier assumption that rank $A=m$ seems to be essential in the above proof (in order to choose B), in fact, this assumption is not necessary. For if A is totally unimodular and rank $A=r<m$, then any submatrix C of A consisting of r linearly independent rows of A is also totally unimodular, and the matrix A can be replaced by C in the proof.

The integer programming problems that we shall formulate to solve the chessboard puzzles have the equations of (6) replaced by the inequality constraints:

$$AX\leqslant b, \quad X\geqslant 0. \tag{8}$$

For integer programming problems of form (8) it can be shown that the total unimodularity of A is not just sufficient (as in Theorem 1) but is also necessary for every basic solution to be integer.

THEOREM 2. *The matrix A is totally unimodular if and only if every basic solution of* (8) *is integer.*

For the proof, see section 3.3 of Garfinkel and Nemhauser [7].

We have now uncovered a condition which allows us to solve an integer programming problem as a linear programming problem. If A is totally unimodular the optimal linear programming solution will be integer.

The question now is, how do we tell whether a given constraint matrix A is totally unimodular? Naturally its entries must be all 0, 1, and -1, but this property is not sufficient. The following result turns out to be helpful in deciding whether A is totally unimodular or not.

THEOREM 3. *A matrix $A=(a_{ij})$ is totally unimodular if the rows of A can be partitioned into two disjoint sets R_1 and R_2 such that*:

(i) *Every column of A contains at most two nonzero entries.*
(ii) *Every entry in A is* 0, 1, *or* -1.
(iii) *If a_{ik} and a_{jk} $(i \neq j)$ are nonzero and have the same sign, then row i is in R_1 and row j is in R_2 or row i is in R_2 and row j is in R_1.*
(iv) *If a_{ik} and a_{jk} $(i \neq j)$ are nonzero and have different signs, then row i and row j are both in R_1 or both in R_2.*

Proof. We use induction on the size of the submatrices of A. To begin, if A satisfies (i)–(iv), it is obvious that any submatrix of exactly one element is totally unimodular. Consider an arbitrary square submatrix B of size $k+1$. If B has a column of all zeros, it will be singular. If B has a column with exactly one nonzero entry, then its determinant can be expanded about that column and the result follows because of the induction hypothesis on all submatrices of size k. Suppose B has two nonzero entries in every column. Then (iii) and (iv) in the statement of the theorem imply that

$$\sum_{i \in R_1} a_{ik} = \sum_{i \in R_2} a_{ik}, \quad \text{for all } k.$$

This means that $\det B = 0$ since this equation implies that a linear combination of rows is zero.

We now introduce a succinct notation for describing the location of pieces placed on the chessboard. Suppose we have placed a number of pieces of one type on the board. Then the board can be represented by an 8×8 matrix, $X=(x_{ij})$ of zeros and ones, where

$$x_{ij} = \begin{cases} 1 & \text{if the square on the } i\text{th rank and } j\text{th file is occupied,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the solution we found to the rooks' problem given in FIGURE 3(a), can be denoted by:

$$x_{ii} = 1, \quad i = 1,2,\ldots,8,$$
$$x_{ij} = 0, \quad i \neq j.$$

We can now demonstrate how 0-1 integer programming can be used to analyze the chessboard puzzles. We begin with the rooks' problem. Consider the ith rank. There can be no more than one rook placed on it, hence $x_{i1} + x_{i2} + \cdots + x_{i8} \leqslant 1$, $i = 1, 2, \ldots, 8$. Similarly for the jth column, $x_{1j} + x_{2j} + \cdots + x_{8j} \leqslant 1$, $i = 1, 2, \ldots, 8$. Thus we have a family of 16 constraints; one for each of the eight rows and one for each of the eight columns. The objective is to maximize the number of rooks placed on the board. Thus the rooks' problem can be formulated as:

$$\text{Maximize} \qquad \sum_{i=1}^{8} \sum_{j=1}^{8} x_{ij}$$

$$\text{subject to the constraints:}\quad \begin{aligned} &\sum_{j=1}^{8} x_{ij} \leqslant 1, \quad i = 1,2,\ldots,8,\\ &\sum_{i=1}^{8} x_{ij} \leqslant 1, \quad j = 1,2,\ldots,8,\\ &x_{ij} = 0 \text{ or } 1, \quad \text{all } i, j. \end{aligned} \tag{9}$$

In the case of the bishops' puzzle, each nontrivial diagonal (with more than one square) can have no more than one bishop placed on it. As an example, the constraint corresponding to this fact for the leading diagonal is:

$$x_{11} + x_{22} + \cdots + x_{88} \leqslant 1.$$

Thus the bishops' problem can be formulated as:

$$\begin{aligned} &\text{Maximize} && \sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij}\\ &\text{subject to the constraints:} && \sum_{i+j=k} x_{ij} \leqslant 1, \quad k = 3,4,\ldots,15,\\ &&& \sum_{i-j=k} x_{ij} \leqslant 1, \quad k = -6,-5,\ldots,6,\\ &&& x_{ij} = 0 \text{ or } 1, \quad \text{all } i, j. \end{aligned} \tag{10}$$

As the queen's move is a combination of the rook's and bishop's moves, it is possible to formulate the puzzle with the constraint set being the totality of the rook's and bishop's constraints: (9) and (10).

We now examine the kings' puzzle. We must ensure that if a king is placed on a square s of rank i and file j, then no kings are placed on the squares touching a side or corner of it. This can be done by allowing no more than one king in every block for four squares. (Here we define a block of four squares, with square (i, j) (or rank i and file j) in the top left-hand corner, to be the squares: (i, j), $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$.) Since there can be no more than one king present in the block we have:

$$x_{ij} + x_{i+1,j} + x_{i,j+1} + x_{i+1,j+1} \leqslant 1.$$

Thus the kings' problem can be formulated as:

$$\begin{aligned} &\text{Maximize} && \sum_{i=1}^{8}\sum_{j=1}^{8} x_{ij}\\ &\text{subject to the constraints:} && \sum_{k=0}^{1}\sum_{m=0}^{1} x_{i+k,j+m} \leqslant 1, \quad i = 1,2,\ldots,7,\ j = 1,2,\ldots,7,\\ &&& x_{ij} = 0 \text{ or } 1, \quad \text{all } i, j. \end{aligned}$$

The knights' puzzle is a little more awkward because of the irregularity of the knight's move. If a knight is placed on a square with rank i and file j, we can identify all squares which are a knight's move away from (i, j) and pair each such square with (i, j). Thus we build up all pairs of squares which are a knight's move away from each other. In order to guarantee that we do not duplicate pairs we consider only ordered pairs of squares $[s, s']$, where s is to the left of s'. Thus if s is the square (i, j), the following ordered pairs will be formed:

$$[(i,j),\ (i-1, j+2)], [(i,j),\ (i+1, j+2)],$$
$$[(i,j),\ (i-2, j+1)], [(i,j),\ (i+2, j+1)].$$

Since a knight can occupy at most one of the two squares of any pair, the knights' puzzle can be formulated as:

$$\text{Maximize} \quad \sum_{i=1}^{8} \sum_{j=1}^{8} x_{ij}$$

$$\begin{array}{lll}
\text{subject to the constraints:} & x_{ij} + x_{i-1,j+2} \leqslant 1, & i = 2,3,\ldots,8,\ j = 1,2,\ldots,6, \\
& x_{ij} + x_{i+1,j+2} \leqslant 1, & i = 1,2,\ldots,7,\ j = 1,2,\ldots,6, \\
& x_{ij} + x_{i-2,j+1} \leqslant 1, & i = 3,4,\ldots,8,\ j = 1,2,\ldots,7, \\
& x_{ij} + x_{i+2,j+1} \leqslant 1, & i = 1,2,\ldots,6,\ j = 1,2,\ldots,7, \\
& x_{ij} = 0 \text{ or } 1, & \text{all } i, j.
\end{array}$$

The coefficient matrices A for the system of constraints for the puzzles for 3×3 boards are given in FIGURE 4. It can be seen that matrices (a) and (b) are totally unimodular by using Theorem 3; the first three rows comprise the set R_1 and the second three rows comprise the set R_2 in each case. (In the $n \times n$ puzzles for rooks and bishops the first n rows of the coefficient matrix comprise R_1 and the second n rows form R_2.)

The conditions of Theorem 3 certainly do not hold for matrices (c) and (d). However the conditions stated in Theorem 3 are *sufficient* rather than necessary and these matrices and their $n \times n$ counterparts are totally unimodular.

Theorem 3 can be used to show that matrix (e) is totally unimodular. However the 3×3 board puzzle for knights is a special case and coefficient matrices for the knights' problem on larger boards do not satisfy the conditions. We notice, though, that for larger boards, the coefficient matrix A always has the property that each row has exactly two unit entries. If we divide the rows

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}$$

(a)

$$\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

(b)

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

(c)

$$\begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}$$

(d)

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

(e)

FIGURE 4. Coefficient matrices for the system of constraints for the various puzzles using chess pieces on a 3×3 chessboard. (a) rooks, (b) bishops, (c) queens, (d) kings, and (e) knights.

into two sets, the odd-numbered rows and the even-numbered rows, we can show (via Theorem 3) that A^T is totally unimodular. A theorem in [**18**] states: *A^T is totally unimodular implies that A is totally unimodular.* Hence the coefficient matrix A for the knights' puzzle is totally unimodular for any sized board.

We close with a solution to the rooks' problem (9). For brevity, we consider the 3×3 case whose coefficient matrix A is in FIGURE 4(a). Although this is a 0-1 integer programming problem, we know that since A is totally unimodular, we can solve it by linear programming. We use the simplex method as explained in Taha [**23**].

The problem is first transformed into standard form by the introduction of slack variable x_i into the ith constraint, $1 \leqslant i \leqslant 6$. Let a_α denote the column vector corresponding to variable x_α. The initial simplex tableau, with the objective function represented in the bottom row is:

a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}	a_1	a_2	a_3	a_4	a_5	a_6	b
(1)	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	1	1	1	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	1	1	0	0	1	0	0	0	1
1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	1
0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	1
0	0	1	0	0	1	0	0	1	0	0	0	0	0	1	1
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0

There are a number of ties for selection of the pivot element. This occurs at each iteration, and we illustrate just one combination here. (Each combination of tie-breaking yields a different solution.) The rest of the iterations follow (pivot elements are circled):

a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}	a_1	a_2	a_3	a_4	a_5	a_6	b
1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	1	(1)	1	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	1	1	0	0	1	0	0	0	1
0	−1	−1	1	0	0	1	0	0	−1	0	0	1	0	0	0
0	1	0	0	1	0	0	1	0	0	0	0	0	1	0	1
0	0	1	0	0	1	0	0	1	0	0	0	0	0	1	1
0	0	0	1	1	1	1	1	1	−1	0	0	0	0	0	−1

a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}	a_1	a_2	a_3	a_4	a_5	a_6	b
1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	1	1	1	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	1	(1)	0	0	1	0	0	0	1
0	−1	−1	1	0	0	1	0	0	−1	0	0	1	0	0	0
0	1	0	−1	0	−1	0	1	0	0	−1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	0	0	0	1	1
0	0	0	0	0	0	1	1	1	−1	−1	0	0	0	0	−2

a_{11}	a_{12}	a_{13}	a_{21}	a_{22}	a_{23}	a_{31}	a_{32}	a_{33}	a_1	a_2	a_3	a_4	a_5	a_6	b
1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	1	1	1	0	0	0	0	1	0	0	0	0	1
0	0	0	0	0	0	1	1	1	0	0	1	0	0	0	1
0	−1	−1	1	0	0	1	0	0	−1	0	0	1	0	0	0
0	1	0	−1	0	−1	0	1	0	0	−1	0	0	1	0	0
0	0	1	0	0	1	−1	−1	0	0	0	−1	0	0	1	0
0	0	0	0	0	0	0	0	0	−1	−1	−1	0	0	0	−3

The final tableau depicts the 3×3 analogue of the solution illustrated in FIGURE 3(a):

$$x_{ii}, \quad i = 1,2,3,$$

with all other variables zero.

The authors would like to express their sincere thanks to the editor and an anonymous reviewer whose comments greatly improved the paper.

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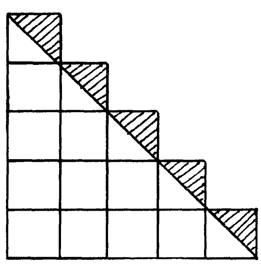
Proof without words: Sum of integers

$$1+2+3+\cdots+n=\frac{n^2}{2}+\frac{n}{2}$$



—IAN RICHARDS
University of Minnesota
Minneapolis, MN 55455

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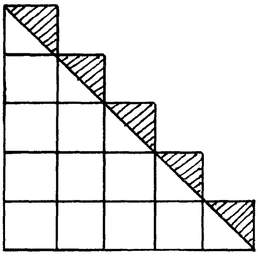


—IAN RICHARDS
University of Minnesota
Minneapolis, MN 55455

Guess a Number—with Lying

JOEL SPENCER
SUNY at Stony Brook
Stony Brook, NY 11794

What's green, hangs on a wall, and whistles?
I don't know.
A red herring.
But it's not green.
You can paint it green.
But it doesn't hang on the wall.
You can hang it on the wall.
But it doesn't whistle.
So, it doesn't whistle.

Stainislas Ulam, in his autobiography *Adventures of a Mathematician*, raises an interesting question (italics ours):

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: Is the number in the first half-million? and then again reduce the reservoir of numbers in the next question by one-half, and so on. Finally the number is obtained in less than $\log_2(1000000)$. *Now suppose one were allowed to lie once or twice, then how many questions would one need to get the right answer?*

A number of technical papers [**1**], [**2**] have explored aspects of this problem. Although we discuss some possible answers to Ulam's question, our main emphasis will be on the actual play of the game. We shall assume the first player (called the **Responder**) is allowed to lie at most once. The Responder selects a number x between 1 and n and the second player (the **Questioner**) is allowed k questions. All questions must be of the form: Is $x > a$? After k rounds the Questioner wins if he knows, with proof, the number x. We further allow the Responder to play a "Devil's stategy." By this we mean that the Responder does not actually think of a number x before the game begins but only responds in a consistent manner, that is, at all times there is some x for which he has lied at most once. The reader is urged to try a few games before proceeding. (The values $n = 100$, $k = 11$ make for interesting play.)

We observe that when no lies are permitted the game has an exact solution. If $n \leqslant 2^k$ then the Questioner has a win by the standard halving strategy. If $n > 2^k$ then the Responder has the Devil's strategy (that clever ninth graders occasionally discover) of answering each question so that the reservoir of numbers is at least half of what it was. After k questions there will be more than $n2^{-k}$, hence at least two, numbers remaining.

Let us discuss some possible strategies for the Questioner when one lie is allowed. Suppose that with no lies permitted, u questions suffice to determine the answer. With one lie allowed, we may wish to ask each question twice. If two consistent answers to the same question are not given (this occurs at most once) repeating the same question a third time will reveal the truth. With this strategy, $2u + 1$ questions are sufficient to determine the answer. A modification of the strategy, discovered by M. A. Spencer, does substantially better when u is large. Questions are asked as if there were no lies in u/a groups of a questions. After each group of questions, two further questions are asked to confirm the previous answers. If confirmation of previous answers is not received, which occurs at most once, all $(a + 2)$ answers are thrown out. The lie has been exposed and the Questioner continues with the standard halving strategy.

To illustrate this strategy, suppose $n = 1{,}000{,}000$ and $a = 6$. First six rounds: *Is* $x > 500{,}000$? No. *Is* $x > 250{,}000$? Yes. *Is* $x > 375{,}000$? Yes. *Is* $x > 437{,}500$? No. *Is* $x > 406{,}250$? Yes. *Is* $x > 421{,}875$? Yes. These would be followed by: *Is* $x > 421{,}875$? *Is* $x > 437{,}500$? If the answers are Yes followed by No then the Questioner knows that $421{,}875 < x \leqslant 437{,}500$.

The total number of questions required to determine the number x using this strategy is at most approximately $(u/a)(a+2)+(a+2)$. We set $a \sim \sqrt{2u}$ to minimize this expression so that the Questioner requires at most approximately $u + 2\sqrt{2u} + 2$ rounds to determine the number. This strategy, though suboptimal, is extremely simple to implement.

The sample game shown in FIGURE 1 with $n = 100$ and $k = 11$ shall provide a basis for our further discussion. The answer to the fifth question in that game exposes a lie, though at that point we do not know whether the lie is in response to the first or the fifth question. The game is then reduced to finding a number between 46 and 100 (inclusive) and the normal halving strategy described by Ulam produces the number 65 after the final question and answer.

It is this author's personal experience that the above example is typical of actual play. The Responder makes his lie very early. The Questioner attempts to expose a lie by asking precisely the same question more than once. These, however, are observations of psychology and are not reflected in the mathematical analysis of the game.

In order to mathematically analyze the game, and attempt to answer Ulam's query, we shall imitate the analysis of the game with no lies permitted. The key difference is that we define a **possibility** as an ordered pair (x, L) where x is the number chosen and $L, 0 \leqslant L \leqslant k$, is the number of the question to which the Respondent *lies*. (If $L = 0$, the Responder does not lie.) For $2^k < n(k+1)$ (for example, $n = 100$, $k = 10$) there is a Devil's strategy. To each question the answers Yes and No split the possibilities into two disjoint classes. The Responder gives the answer that leaves the larger class. After k rounds there will remain at least two possibilities. But (and this is essential) these possibilities cannot have the same number x, for if the Questioner had determined the number x he would know, by checking previous answers, to which question the Responder had lied.

We illustrate the Devil's strategy with the situation in our sample game for $n = 100$, $k = 11$ when question 3 is asked but not yet answered. The set of possibilities is:

(x, L)	$0 < x \leqslant 25, L = 2$	25 possibilities
	$25 < x \leqslant 50, L \neq 1,2$	250 possibilities
	$50 < x \leqslant 100, L = 1$	50 possibilities

for the total of 325 possibilities. The third question: Is $x > 38$? splits the above set as follows:

NO CLASS		YES CLASS	
$0 < x \leqslant 25, L = 2$	25	$25 < x \leqslant 38, L = 3$	13
$25 < x \leqslant 38, L \neq 1,2,3$	$13 \times 9 = 117$	$38 < x \leqslant 50, L \neq 1,2,3$	$12 \times 9 = 108$
$38 < x \leqslant 50, L = 3$	12	$50 < x \leqslant 100, L = 1$	50
	154		171

and so the proper Devil's strategy is to answer Yes.

The Questioner's strategy is to select a question that will balance the Yes Class and the No Class as evenly as possible. In the above situation if the question "Is $x > 38$?" is adjusted to "Is $x > 39$?", the No Class gains the possibilities $(39, L)$, $L \neq 1,2,3$ and loses $(39,3)$ for a net gain of eight, making the No Class/Yes Class split 162/163. Thus "Is $x > 39$?" is the proper third question.

When it is the Questioner's turn let us call the set of remaining possibilities the **state** and the number of remaining possibilities the **weight** of the state. The typical state may be written in the form $S^iM^jS^m$ where S^i represents i consecutive numbers x which satisfy all answers but one and

1.	Is $x > 50$?	No.
2.	Is $x > 25$?	Yes.
3.	Is $x > 38$?	Yes.
4.	Is $x > 45$?	Yes.
5.	Is $x > 50$?	Yes.
6.	Is $x > 72$?	No.
7.	Is $x > 59$?	Yes.
8.	Is $x > 66$?	No.
9.	Is $x > 63$?	Yes.
10.	Is $x > 65$?	No.
11.	Is $x > 64$?	Yes.

FIGURE 1. **Sample game with $n = 100$, $k = 11$.**

M^j represents j consecutive numbers x which satisfy all answers. We'll call the numbers S^i a side group and the numbers M^j the main group. In the sample game, the initial state M^{100} becomes successively: $M^{50}S^{50}$, $S^{25}M^{25}S^{50}$, $S^{13}M^{12}S^{50}$, $S^7M^5S^{50}$ and, after the fifth response, S^5S^{50}. (For example, after the third response, $\{39,\ldots,50\}$ is the main group and $\{26,\ldots,38\}$, $\{51,\ldots,100\}$ are the side groups.) If there are t questions remaining, the state $S^iM^jS^m$ has weight $w = i + (t+1)j + m$.

With a little practice, the Questioner can rapidly decide (at least within one number) the appropriate question. Let the state be $S^iM^jS^m$ with t questions remaining. Let a_0 lie in the center of the main group. (If lies were not allowed, then "Is $x > a_0$?" would be the proper question.) The side groups force an adjustment of $(m-i)/(t-1)$ and the Questioner should ask "Is $x > a_0 + (m-i)/(t-1)$?". (Note, roughly, that as t decreases, i.e., as the game nears its conclusion, the influence of the side groups becomes stronger.) If the number $a_0 + (m-i)/(t-1)$ is not in the main group then this method does not apply. Let the state be $S^iM^jS^m$ with $m > i$ (the other case being symmetric) and let E be the largest number in the right side group. If w denotes the weight, the Questioner then asks "Is $x > a$?" where $a = E - (w/2) + j$. For example, after the fourth round in our sample game the state is $S^7M^5S^{50}$ with seven questions remaining ($t = 7$). Here $w = 97$, $E = 100$ and the Questioner asks: "Is $x > 57$?". If Yes is the response, then the new state is S^5S^{43}, where the side groups are the integers from 46 to 50 and from 58 to 100. The Questioner should then follow the normal halving strategy (bearing in mind that the median is no longer the average of the extremes).

For which n, k does this "even splitting" strategy lead to a win for Questioner? A precise answer to this question is difficult because it is not always possible to split the set of possibilities evenly. For example, in our sample game, the state after the first question and answer is $M^{50}S^{50}$ and the weight is $w = 50(11) + 50(1) = 600$. The question "Is $x > 28$?" gives a No Class/Yes Class split of 293/307 and the question "Is $x > 29$?" gives a No Class/Yes Class split of 302/298. This leads us to an analysis of how closely the possibilities may be split.

Suppose that $i-1$ questions have already been asked and answered and that the current state has weight v_{i-1}. A question "Is $x > a$?" will split the v_{i-1} possibilities into a No Class and a Yes Class. Let $f(a)$ be the size of the No Class. The Questioner seeks an a such that $f(a)$ is as close as possible to $v_{i-1}/2$. (For example, in our sample game after two questions had been asked and answered, the Question "Is $x > 38$?" would induce a No Class/Yes Class split of 154/171. There $v_2 = 325$ and $f(38) = 154$.) If the question "Is $x > a-1$?" is changed to "Is $x > a$?" where a is in the main group then the No Class gains the possibilities (a, L), $L \neq 1,\ldots,i$ and loses (a, i) for a net gain of $k - i$. In this case $f(a) = f(a-1) + (k-i)$. (In the sample game, $f(39) = 162$.) When a is in a side group, $f(a) = f(a-1) + 1$ and, of course, when a is not in any group, $f(a) = f(a-1)$. Assume $i \leqslant k-1$. The function f satisfies $f(0) \leqslant v_{i-1}/2$ and $f(n) \geqslant v_{i-1}/2$ (why?) and has jumps of at most $k-i$. Hence for some a, $f(a)$ is within $(k-i)/2$ of $v_{i-1}/2$. (This may be considered a discrete version of the Mean Value Theorem. If, for example, a function goes from less than 162.5 to more than 162.5 with jumps of at most 8 then at some point its value is within 4 of 162.5.) The Questioner asks "Is $x > a$?" for that a. *Regardless* of the answer, the new weight v_i satisfies

$$v_i \leqslant v_{i-1}/2 + (k-i)/2. \tag{1}$$

To analyze inequality (1), we set $w_i = v_i 2^{i-k}$ so that

$$w_i \leqslant w_{i-1} + (k-i)2^{i-k-1} \tag{2}$$

and hence

$$w_{k-1} \leqslant w_0 + \sum_{i=0}^{k-1} (k-i)2^{i-k-1}. \tag{3}$$

The term $(k-i)2^{i-k-1}$ in (2) is essentially the effect of the unevenness of the splitting on the ith question. The terms become significant only when $k-i$ is small. Setting $j = k-i$, we obtain

$$\sum_{i=0}^{k-1} (k-i)2^{i-k-1} = \sum_{j=1}^{k} j2^{-j-1}. \tag{4}$$

The right hand side of (4) is bounded by the infinite sum $\sum j2^{-j-1}$ which converges to 1 (a nice exercise!). Let $v_0 = n(k+1)$, the initial number of possibilities. Then $w_0 = n(k+1)2^{-k}$ and from (3) and (4) it follows that

$$w_{k-1} < n(k+1)2^{-k} + 1$$

and thus

$$v_{k-1} = 2w_{k-1} < 2n(k+1)2^{-k} + 2.$$

Assume that $n \leqslant 2^{k-1}/(k+1)$. The Questioner, applying the halving strategy, can assure $v_{k-1} < 3$. With one question remaining, the state is either M, S, or SS. In the first two cases, the number x has already been determined. In the third case, the lie has already been exposed and the number x may be determined with the last question.

Combining these observations, we obtain the following results for this strategy.

(i) If $n \leqslant 2^{k-1}/(k+1)$, the Questioner wins,
(ii) If $n > 2^k/(k+1)$, the Responder wins.

For example, if $n = 100$, the Responder wins with 10 questions and the Questioner wins with 12 questions. What happens if there are 11 questions? A detailed study (using a small computer) of endgames provided an answer. Checking all states with five questions remaining, it was found that if $v_{k-5} < 26$ then the Questioner has a winning strategy. ($S^4M^3S^4$ is the minimal weight state from which the Responder wins.) A modification of our analysis of inequality (1) can be used to show

$$v_{k-5} < v_0 2^{-k+5} + 6.$$

Thus for this case we can improve (i) to

(iii) If $n \leqslant \frac{5}{8}2^k/(k+1)$, the Questioner wins ($k \geqslant 5$).

For any value of n, the formulae (ii), (iii) determine the required number of questions within one. A more detailed endgame study would certainly increase the constant 5/8 in (iii). But it seems very difficult to determine whether the answer to Ulam's original problem is twenty-five or twenty-six.

References

[1] T. A. Brylawski, The Mathematics of Watergate (An Analysis of a Two-Party Game), Univ. of N. Carolina, 1978.

[2] D. J. Kleitman, A. R. Meyer, R. L. Rivest, J. Spencer, K. Winklmann, Coping with errors in binary search procedures, J. Comput. System Sci., 20 (1980) 396–404.

[3] S. M. Ulam, Adventures of a Mathematician, Scribner, New York, 1976.

PROBLEMS

LEROY F. MEYERS, Editor
G. A. EDGAR, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before August 1, 1984.

1186. (a) Show how to arrange the 24 permutations of the set $\{1,2,3,4\}$ in a sequence with the property that adjacent members of the sequence differ in each coordinate. (Two permutations (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) differ in each coordinate if $a_i \neq b_i$ for $i = 1,2,3,4$.)

*(b) For which n can the $n!$ permutations of the integers from 1 through n be arranged in a similar manner? [*Stanley Rabinowitz, Merrimack, New Hampshire.*]

1187. Let the chord AB of circle O be trisected at C and D. Let P be any point on the circle other than A and B. Extend the lines PD and PC to intersect the circle in E and F, respectively. Extend the lines EC and FD to intersect the circle in G and H, respectively. Let GF and HE intersect AB in L and M, respectively. Prove that $AL = BM$. [*R. S. Luthar, University of Wisconsin Center, Janesville.*]

1188. Prove that for all real x and all integers $n > 1$,

$$|\cos(2x)|^{n/2} \leqslant |\cos^{2n}x - \sin^{2n}x|.$$

When does equality hold? [*Vania D. Mascioni, student, ETH Zürich, Switzerland.*]

1189. $.8^2 + .6^2 = 1$. How many other pairs of positive real numbers x, y are there whose decimal expansions contain only even digits and satisfy $x^2 + y^2 = 1$? [*James Propp, student, Cambridge University.*]

1190. (a) If abc is a three-digit number in base ten, where $a > c$, and if $def = abc - cba$ is always considered a three-digit number (even when $d = 0$), then it is known that $def + fed = 1089$. Generalize this result to any base $k > 1$.

(b) For which base(s) is $def + fed$ a perfect square? [*Robert L. Bernhardt, East Carolina University.*]

ASSISTANT EDITORS: DANIEL B. SHAPIRO and WILLIAM A. MCWORTER, JR., *The Ohio State University.*

*We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) will be placed next to a problem number to indicate that the proposer did not supply a solution.*

Solutions should be written in a style appropriate for Mathematics Magazine. Each solution should begin on a separate sheet containing the solver's name and full address. It is not necessary to submit duplicate copies.

Send all communications to the problems department to Leroy F. Meyers, Mathematics Department, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.

Quickie

Solution to the Quickie appears on page 115.

Q688. Show that

$$\sqrt{a^2+b^2+c^2}+\sqrt{b^2+c^2+d^2}+\sqrt{c^2+d^2+a^2}+\sqrt{d^2+a^2+b^2}\geqslant 3\sqrt{a^2+b^2+c^2+d^2}.$$

[*M. S. Klamkin, University of Alberta.*]

Solutions

A Positive, Increasing Function — November 1982

1160. In their "completion" of my solution to problem 1129 [pp. 304–305, Nov. 1982], the editors claim: "Elementary but tedious calculation shows that the function f defined by

$$f(x)=\frac{1}{\sin x}-\frac{1}{x},\quad 0<x\leqslant\frac{\pi}{2},$$

is positive and increasing." Justify the claim. [*Anon, Erewhon-upon-Spanish River.*]

Solution: It is well known that the power series of $\csc x - x^{-1}$ converges on $(0, \pi/2)$ and has all its coefficients positive. (See, for example, Knopp, *Theory and application of infinite series*, pp. 204, 237.) Hence f is positive, increasing, and somewhat more.

For a more strictly elementary proof, note that $f(x)>0$ since $x>\sin x$ for $x>0$, and consider

$$f'(x)=\frac{\sin^2 x - x^2\cos x}{x^2\sin^2 x}.$$

This is positive if $\sin^2 x > x^2\cos x$, which is true by Mitrinović, *Analytic inequalities*, no. 3.4.18, p. 238. Here is the proof.

Let $g(x)=x-(\sin x)(\cos x)^{-1/2}$. Then

$$g''(x)=\tfrac{1}{4}(\sin x)(\cos x)^{-5/2}(\cos^2 x-3).$$

Obviously $g''(x)<0$ for $x\in(0,\pi/2)$, so $g'(x)<g'(0)=0$ and $g(x)<g(0)=0$.

Mitrinović shows that even $x^{-3}\sin^3 x>\cos x$. (Cf. problem 1137, this MAGAZINE, vol. 56 (1983), pp. 53–54.)

R. P. BOAS
Northwestern University

Also solved by Duane M. Broline, F. S. Cater, Curtis Cooper, Thomas P. Dence, Thomas E. Elsner, Enzo R. Gentile (Argentina), Leon Gerber, G. A. Heuer, Hans Kappus (Switzerland), the late H. Kestelman (England, two solutions), M. S. Klamkin (Canada), Benjamin G. Klein, L. Kuipers (Switzerland), Vania D. Mascioni (student, Switzerland), Peter L. Montgomery, Roger B. Nelsen, William A. Newcomb, William Noble, M. Ratnaprabhu & T. Narasimham (India), Daniel A. Rawsthorne, St. Olaf College Problem Solving Group, Heinz-Jürgen Seiffert (student, West Germany), Robert S. Stacy (West Germany), J. Suck (West Germany, two solutions), Gerald Thompson & Marlin Brown, W. R. Utz, S. K. Venkatesan (India), Michael Vowe (Switzerland), Edward T. H. Wang (Canada), Michael Woltermann, and the proposer (three solutions). There were nine incorrect or seriously incomplete solutions.

Not all of the calculations were tedious. Many solvers stated that

$$x^2(\sin^2 x)f'(x) = \sin^2 x - x^2\cos x > \left(x - \frac{1}{6}x^3\right)^2 - x^2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) = \frac{1}{72}x^4(12 - x^2) > 0$$

for $x^2 < 12$, forgetting that squaring both sides of the inequality $\sin x > x - \frac{1}{6}x^3$ works only for $x^2 < 6$ (or slightly beyond), but certainly for $x^2 < (\pi/2)^2$.

Cater showed that $1/6$ is the largest constant k for which $f(x) - kx$ is positive and increasing on $(0, \pi)$.

Ring with Unique Nonzero Nondivisor of Zero — January 1983

1162. Suppose R is a finite associative ring in which there is exactly one nonzero element which is neither a left nor a right divisor of zero. Show that R is Boolean, i.e., that $x^2 = x$ for all $x \in R$. [*Enzo R. Gentile, Buenos Aires, Argentina.*]

Solution I: Let e be the unique element which is neither a left nor a right divisor of zero. Since neither e^2 nor the additive inverse of e is a divisor of zero, $e^2 = e$ and $e + e = 0$. Also, from

$$e(ex - x) = ex - ex = 0$$

it follows that $ex = x$. Similarly, $xe = x$. Hence e is an identity for R. Furthermore, R is of characteristic 2. If $x^2 = 0$, then

$$(x + e)^2 = x^2 + e^2 = e.$$

Hence $x + e$ is neither a left nor a right divisor of zero. By the uniqueness of e, it follows that $x = 0$.

Choose $x \in R$, $x \neq 0$, and consider $\{x^{2^n} | n \text{ a nonnegative integer}\}$. Since R is finite, there is a least nonnegative integer r such that $x^{2^r} = x^{2^s}$, for $r < s$. If $r > 0$, then $x^{2^{r-1}} \neq x^{2^{s-1}}$, so that $x^{2^{r-1}} + x^{2^{s-1}} \neq 0$, but

$$(x^{2^{r-1}} + x^{2^{s-1}})^2 = x^{2^r} + x^{2^s} = 0,$$

a contradiction. Thus $r = 0$, and so $x = x^{2^s}$ with $s \geqslant 1$. Let $z = e + x + x^{2^s - 1}$. Then

$$xz = x + x^2 + x^{2^s} = x + x^2 + x = x^2 \neq 0.$$

Hence $z \neq 0$. Suppose $z \neq e$. Then there exists $y \in R$, $y \neq 0$, such that either yz or zy is zero. Without loss of generality, suppose $zy = 0$. Then $0 = xzy = x^2y$. Thus, from $2^s - 1 \geqslant 1$ it follows that $xy = x^{2^s - 1}xy = 0$, and

$$0 = zy = y + xy + x^{2^s - 1}y = y,$$

a contradiction. Hence $e = z = e + x + x^{2^s - 1}$. Therefore $x = x^{2^s - 1}$ and

$$x^2 = xx^{2^s - 1} = x^{2^s} = x,$$

as required.

DUANE M. BROLINE
University of Evansville

Solution II: [As in solution I, R has characteristic 2. Write 1 for e.] R is a finite-dimensional vector space over GF(2). Let $y \in R \setminus \mathrm{GF}(2)$. Then there is a monic polynomial $P(x)$ over GF(2) of least degree such that $P(y) = 0$. Let n be the degree of $P(x)$. Then $n \geqslant 2$ because $y \in R \setminus \mathrm{GF}(2)$. If $Q(x)$ is any irreducible polynomial of positive degree less than n, then $Q(y)$ is a linear combination (a sum, actually) of the independent vectors $1, y, \ldots, y^{n-1}$, and involves 1 and at least one of the y^i with $i \geqslant 1$. Hence $Q(y) \neq 1 \in \mathrm{GF}(2)$. On the other hand, $Q(x)$ divides $P(x)$, because otherwise $Q(x)$ and $P(x)$ would be relatively prime and there would exist polynomials $u(x), v(x) \in \mathrm{GF}(2)[x]$ such that

$$Q(x)u(x) + P(x)v(x) = 1.$$

But then $Q(y)u(y)=1$ and $Q(y)$ has an inverse, against the fact that $Q(y) \neq 1$. Hence every irreducible polynomial of degree dividing $n-1$ divides $P(x)$, and so their product $x^{2^{n-1}}-x$ divides $P(x)$. Thus $2^{n-1} \leqslant n$, and so $n=2$ and $P(x)=x^2-x$. Q.E.D.

WILLIAM A. MCWORTER, JR.
The Ohio State University

Solution III: [As in solution I, e is the identity of R, and $x^2=0$ implies $x=0$.] Consequently, R has no nonzero nilpotent elements. Since R is finite, R is the direct sum $\oplus_{i=1}^{n} D_i$ of division rings D_i, by the Wedderburn-Artin theorem. Let e_i denote the identity of D_i. Suppose that there exists $j \in \{1,2,\ldots,n\}$ such that D_j is not isomorphic to $Z/(2)$, the integers modulo 2. Let $x_j \in D_j$ such that $x_j \neq 0$ and $x_j \neq e_j$. Then

$$e_1+e_2+\cdots+e_{j-1}+x_j+e_{j+1}+\cdots+e_n \neq e$$

is a unit of R. Contradiction! Thus each D_i is a copy of $Z/(2)$. Consequently, R is Boolean.

GARY BIRKENMEIER
Southeast Missouri State University

Also solved by Samuel Chort, A. J. Douglas (England), William H. Gustafson, Joel K. Haack, Henry Heatherly, I. E. Leonard III, Paul Peck, A. Pianzola (student, Canada), Dennis Spellman, Luis F. Tenorio (student), Gregory P. Wene, and the proposer. There was one incorrect solution.

Peck noted the similarity with MONTHLY problem 6284, vol. 89 (1982), pp. 135–136. Tenorio used Proposition 2 of the solution to problem 1111, this MAGAZINE, vol. 55 (1982), pp. 300–303. Heatherly remarked that the condition "R is finite" may be replaced by any of the following: R has the descending (ascending) chain condition on left (right) ideals; R has the ascending chain condition on left (right) annihilators and has finite left (right) rank. Chort varied the hypotheses, and showed, for example, that the conclusion follows if there is a unique element which is not a right divisor of zero. The proposer referred to his (Spanish) article in *Notas de Matemáticas* (Lima, Peru), vol. 5 (1967), pp. 103–105.

A One-to-one Continuous Function's Discontinuous Inverse — January 1983

1163*. Prove or disprove: If f is a one-to-one function of R into R and f is continuous at some $b \in R$, then f^{-1}, the inverse of f, is continuous at $f(b)$. [*Michael Grossman, University of Lowell.*]

Solution: Here is a counterexample. Define f by:

$$f(x)=\begin{cases} 1/(2n) & \text{if } x=1/n \text{ for some positive integer } n; \\ 1/n & \text{if } x=n \text{ for some positive odd integer } n>1; \\ n/2 & \text{if } x=n \text{ for some positive even integer } n; \\ x & \text{otherwise.} \end{cases}$$

That is, $f(x)=x$ except that the range values $1,1/2,1/3,\ldots$ have been replaced by $1/2,1/4,1/6,\ldots$, the range values $3,5,7,\ldots$ by $1/3,1/5,1/7,\ldots$, and the range values $2,4,6,\ldots$ by $1,2,3,\ldots$. It is easy to see that f is one-to-one and onto, as well as continuous at 0. However, f^{-1} is not continuous at $f(0)=0$, since $f^{-1}(1/n)=n$ whenever n is an odd positive integer other than 1.

ROBERT E. MEGGINSON, student
University of Illinois

Also solved by Keith Alford, Robert E. Bernstein (three solutions), Robert Blodgett & Jerome Schneidman, Stephan C. Carlson, Julio Castiñeira (Spain), F. S. Cater, James E. Conklin, Roger L. Cooke, A. K. Desai (India), Charles R. Diminnie, Jessie Ann Engle, Enzo R. Gentile (Argentina), Maxim Goldberg (student), Jerrold W. Grossman, Chico Problem Group, Joel K. Haack, Carsten Hansen & Daniel Otero, George C. Harrison & Mou-Liang Kung, G. A. Heuer, Paul Ilacqua, L. R. King, Vania D. Mascioni (student, Switzerland), Robert Messer, Mark D. Meyerson, William A. Newcomb, L. Pintér (Hungary), Daniel A. Rawsthorne, Wulf D. Rehder, Benjamin L. Schwartz, Harry Sedinger, R. A. Struble, Ronald Tannenwald, Luis F. Tenorio (student), and Charles A. Wilson. Four incorrect solutions were received.

Numerals Representing Primes? **January 1983**

1164. Are there any numerals of two or more "digits" which represent primes in all sufficiently large bases? (Obviously, each prime is represented by a single "digit" in every base larger than the prime.) [*A. Joseph Berlau, Hartsdale, New York.*]

Solution: No. Let the a_i represent the "digits" and b the base. Consider $N = \sum_{i=0}^{n} a_i b^i$, $n \geqslant 1$. If $a_0 = 0$, then $b|N$, and so N is composite when b is.

If $a_0 \neq 0$, then there are at least two nonzero "digits": $a_0 \geqslant 1$, $a_n \geqslant 1$. Let $k = \sum_{i=0}^{n} a_i$ and consider $b = kp + 1$ with p a positive integer. Since $b \equiv 1 \pmod{k}$, we have

$$N \equiv \sum_{i=0}^{n} a_i b^i \equiv \sum_{i=0}^{n} a_i \equiv k \equiv 0 \pmod{k}.$$

Hence $k|N$. But $N > a_n b^n \geqslant b^n \geqslant b > k > 1$. Hence N is composite for the infinitely many bases $kp + 1$.

Ada Booth
University of Santa Clara

Also solved by Joel K. Haack, Mark Kantrowitz (student), Mary S. Krimmel, Gary Ling, Vania D. Mascioni (student, Switzerland), William Mixon, Bill Olk (student), Benjamin L. Schwartz, Aristomenis Siskakis, Robert S. Stacy (West Germany), Douglas A. Taylor (student), Douglas H. Underwood, and the proposer. There was one incorrect solution.

Several solvers remarked that the problem should have been a Quickie, since the result follows immediately from the fact that a nonconstant polynomial cannot represent primes for all sufficiently large values of its argument. L. Kuipers (Switzerland) noted that, by Dirichlet's theorem on arithmetic progressions, $a_1 a_0$ represents a prime in infinitely many bases if gcd $(a_1, a_0) = 1$.

Lattice Point Count **March 1983**

1166. Let m and n be nonnegative integers, let c be a positive irrational number, and let $d = 1/c$. Prove that

$$\sum_{j=0}^{[m+nc]} [n+1+(m-j)d] = \sum_{j=0}^{[n+md]} [m+1+(n-j)c],$$

where the brackets denote the greatest integer function. [*Clark Kimberling, University of Evansville.*]

Solution: The result holds even when c and d are rational. To prove it, let $\alpha = m + nc$ and $\beta = n + md = \alpha d$, and consider the set of lattice points

$$S = \left\{(j,k): j \geqslant 0, k \geqslant 0, \text{and } \frac{j}{\alpha} + \frac{k}{\beta} \leqslant 1\right\}.$$

We wish to show that

$$\sum_{j=0}^{[\alpha]} \left[1 + \beta - \frac{j\beta}{\alpha}\right] = |S| = \sum_{k=0}^{[\beta]} \left[1 + \alpha - \frac{k\alpha}{\beta}\right].$$

(The outer equality is exactly the result we are asked to prove.)

Fix $j \geqslant 0$. Then (j, k) is in S for all integers k such that $0 \leqslant k \leqslant \beta - j\beta/\alpha$, of which there are $[1 + \beta - j\beta/\alpha]$ unless $j/\alpha > 1$ (i.e., unless $j > [\alpha]$), in which case there are no suitable values of k. By summing over j, we obtain the first expression for $|S|$. Similar reasoning yields the second expression for $|S|$, and the equality is established.

More generally, we have

$$\sum_{j=0}^{[\alpha]} [1+f(j)] = \sum_{k=0}^{[\beta]} [1+f^{-1}(k)]$$

for any strictly decreasing function f mapping $[0, \alpha]$ onto $[0, \beta]$,

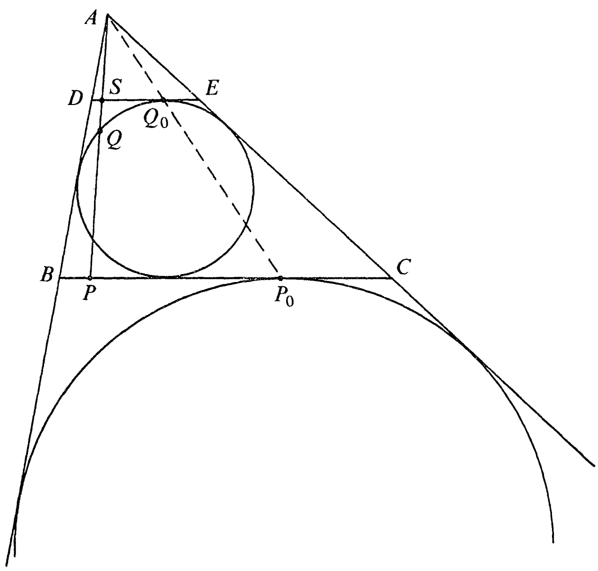
JAMES PROPP, student
Cambridge University

Also solved by Bruce C. Berndt, David Boduch (student), Curtis Cooper, Ray Haertel (Peru), H. G. Mushenheim, Richard Parris, Daniel A. Rawsthorne, C. Ray Rosentrater, and the proposer.

Related problems may be found in Donald E. Knuth, *The art of computer programming*, v. 1 (1969), ex. 1.2.4. 37, 45, 46, pp. 42, 43; solutions, pp. 477, 478.

Incircle and Excircle **March 1983**

1168. Let P be a variable point on side BC of triangle ABC. Segment AP meets the incircle of triangle ABC in two points, Q and R, with Q being closer to A. Prove that the ratio AQ/AP is a minimum when P is the point of contact of the excircle opposite A with side BC. [*Stanley Rabinowitz, Merrimack, New Hampshire.*]



Solution: In the figure, DE is parallel to BC and touches the incircle at Q_0. Consider the magnification (or dilation, or dilatation) with centre A that maps D and E to B and C, respectively; it also maps the incircle (touching DE, AB, AC) to the excircle (touching BC, AB, AC), and maps the point of contact Q_0 to P_0. Hence A, Q_0, P_0 are collinear. Then $AQ_0/AP_0 = AS/AP \leqslant AQ/AP$.

J. F. RIGBY
University College
Cardiff, Wales

Also solved by David Boduch (student), Benny N. Cheng (student, The Philippines), Howard Eves, L. Kuipers (Switzerland), Richard Parris, Harry Zaremba, and the proposer.

1169. Suppose that two players, P_1 and P_2, are engaged in a best-of-three chess tournament, with P_1 playing the white pieces in the first game. Let p_1 be the probability that P_1 wins when playing the white pieces, and let p_2 be the corresponding probability for P_2. Suppose that $0 < p_1 < 1$ and $0 < p_2 < 1$, and that the probability of a draw is 0.

(a) When is it better for P_1 to purposely lose the first game (rather than play to win) if the loser of the first game plays the white pieces for the remaining game(s)?

(b) Show that it is never to P_1's advantage to purposely lose the first game if the rule is that the loser of any game plays the white pieces in the next game. [*Peter Schumer, student, University of Maryland.*]

Solution: (a) If P_1 purposely loses the first game, the probability that P_1 will win the tournament is p_1^2. Otherwise, P_1's probability of winning the tournament is

$$(1-p_1)p_1^2 + p_1(1-p_2^2) = p_1 + p_1^2 - p_1^3 - p_1 p_2^2.$$

Hence P_1 should purposely lose the first game if and only if

$$p_1^2 > p_1 + p_1^2 - p_1^3 - p_1 p_2^2, \quad \text{or} \quad p_1^2 + p_2^2 > 1.$$

Note: In this case it is also advantageous for P_2 to throw the first game, so P_1 should resign on the first move!

(b) If P_1 purposely loses the first game, the probability that P_1 will win the tournament is $p_1(1-p_2)$. Otherwise, P_1's probability of winning the tournament is

$$p_1(1-p_2) + p_1^2 p_2 + (1-p_1)p_1(1-p_2).$$

Clearly, the latter is strictly larger, since $p_1^2 p_2 + (1-p_1)p_1(1-p_2) > 0$,

S. F. BARGER
Youngstown State University

Also solved by R. E. Bernstein, David Boduch (student), Curtis Cooper, Milton P. Eisner, Frederic Gooding, Jr., Raymond N. Greenwell, Joel K. Haack, Mark Kantrowitz (student), Benjamin G. Klein, Rochelle Leibowitz, H. G. Mushenheim, Roger B. Nelsen, Richard Parris, C. Ray Rosentrater, Harry Sedinger, Leon Warman (student), and the proposer. There was one incorrect solution.

Several of the solvers (correctly) compared the probability of winning the tournament having lost the first game with the probability of winning the tournament having won the first game. The proposer noted that the problem is based on a real-life situation.

Answer

Solution to the Quickie which appears on page 110.

Q688.

$$|(a,b,c,0)| + |(0,b,c,d)| + |(a,0,c,d)| + |(a,b,0,d)|$$

$$\geqslant |(a,b,c,0) + (0,b,c,d) + (a,0,c,d) + (a,b,0,d)| = |3(a,b,c,d)|.$$

More generally, let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be vectors with sum $\mathbf{S}$. Then by the triangle inequality,

$$|\mathbf{S} - \mathbf{A}_1| + \cdots + |\mathbf{S} - \mathbf{A}_n| \geqslant (n-1)|\mathbf{S}|.$$

The inequality to be proved corresponds to the special case in which $n = 4$ and the vectors $\mathbf{A}_i$ are mutually orthogonal. There is equality if and only if at least three of the four vectors are null.

REVIEWS

PAUL J. CAMPBELL, Editor
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Kolata, Gina, *Century-old math problem solved*, Science 222 (7 October 1983) 40-41; Mazur, Barry, *Solution of math problem*, *ibid.* (4 November 1983) 456.

The past year has brought in a bumper crop of solutions of long-outstanding mathematical problems. The latest contribution is the resolution of the class number problem for all quadratic imaginary number fields, by D. Zagier (Maryland) and B. Gross (Brown). Such a field consists of numbers of the form $a + b\sqrt{-d}$, for fixed positive integer d and all integers a and b (the article errs in phrasing this, though it does correctly note an exception). A field's class number measures how many ways numbers in the system can be factored into primes of the system, and the class number problem is given to find the largest d whose system has class number k. D. Goldfield (Texas) had made the utterly unexpected connection six years ago of converting the problem into finding a particular kind of elliptic curve, which Gross and Zagier were able to do. Their proof, says Gross, is an equation: "To calculate both sides...takes 100 pages... . It's a mess."

Bergerud, Arthur T., *Prey switching in a simple ecosystem*, Scientific American 249:6 (December 1983) 130-141, 178.

Teachers of differential equations often cite the oscillations of the lynx and Arctic hare populations as a real-life example of predator-prey coupled linear differential equations. Life is never quite so simple as the mathematical model, however, and this article gives the broader context of that biological interaction as it occurs in Newfoundland. The context includes extinction of the timber wolf, introduction of the snowshoe hare, and lynxes periodically switching to caribou. The observations were confirmed by controlled experiments on smaller islands. Students will value learning about the biology, and enjoy trying to model successfully the interactions of the larger number of species actually involved.

Altman, Douglas G., *Statistics in medical journals*, Statistics in Medicine 1 (1982) 59-71.

Even students in introductory statistics can find errors of statistical treatment in such medical journals as the New England Journal of Medicine. This article illustrates the sorts of errors that prevail in medical journals, notes causes and effects of the poor quality of statistics, and suggests how to raise standards (e.g., *statistical* refereeing of papers). Other reviews of the quality of statistics in the published medical literature are cited. Presenting actual examples of mistakes in class (and as exercises) may help students gain confidence and avoid similar errors themselves.

Norwood, Rick, *In abstract terrain: like Caesar's Gaul, all mathematics is divided into three parts*, The Sciences 22 (1982) 12-18.

The three parts are algebra, analysis, and topology. "There are also a few islands--number theory and set theory, for example--and two vast continents that have broken off from the mainland and are drifting out to sea: computer science and statistics." The author, a professor of mathematics, goes on to give an excellent popular account of major achievements in the last five years in each "part": classification of finite groups and public key cryptography; Deligne's advance on the Riemann hypothesis; the Four-Color Theorem, proofs of the Smith conjecture and complements conjecture (mentioned only), Freedman's proof of the Poincaré conjecture for dimension 4, and Cohen's 1982 result (an n-manifold can be immersed in Euclidian space of dimension $2n$ minus the number of 1's in a binary expansion of n). Share the good news.

Loeb, Arthur L., *Kinship graphs--a new representation of dynastic relations*, Perspectives in Computing: Applications in the Academic and Scientific Community 3:2 (May 1983) 28-45.

Use of graphs, stored conveniently in a computer and accessed interactively, simplifies analysis of kinship relations in medieval Burgundy.

Benbow, Camilla Persson, and Stanley, Julian C., *Sex differences in mathematical reasoning ability: more facts*, Science 222 (2 December 1983) 1029-1031.

"...[B]y age 13 a large sex difference in mathematical reasoning ability exists and that it is especially pronounced at the high end of the distribution... ." The latter point is the subject of this paper; in 1980 they compared mean scores on SAT-Math for youth in the Johns Hopkins talent search *ibid.* [12 December 1980] 1262-1264. Again, they conclude that neither environmental influences nor differences in course-taking can account for their result that "males dominate the highest ranges of mathematical reasoning ability before they enter adolescence."

Srinivasan, Bhama, and Sally, Judith (eds.), Emmy Noether in Bryn Mawr, Springer-Verlag, 1983; viii + 182 pp.

This volume contains the proceedings of a symposium honoring the 100th anniversary of the birth of Emmy Noether. The addresses include technical reviews of her research in various areas (together with ensuing developments), plus personal reminiscences and historical remarks. A complete bibliography of her work is appended.

Burckhardt, J.J., *et al.* (eds.), Leonhard Euler 1707-1783: Beiträge zu Leben und Werk, Birkhäuser, 1983; 555 pp., $29.95.

Splendid and thorough memorial volume, offering broad coverage of the vast scope of Euler's work. The many illustrations include a color reproduction of Handmann's pastel portrait shown in this *Magazine*, November 1983, p. 261. That issue (pp. 272-273) contains the full table of contents of this remarkable book.

Smith, Steven B., The Great Mental Calculators: The Psychology, Methods, and Lives of Calculating Prodigies, Past and Present, Columbia U Pr, 1983; xxviii + 374 pp, $25.

Thorough and definitive account of calculating prodigies and their art, including psychology, methods and vignettes of individuals and their accomplishments.

Pais, Abraham, "Subtle is the Lord...": The Science and Life of Albert Einstein, Oxford U Pr, 1982; xvi + 552 pp, $12.95 (P).

Award-winning personal and scientific biography, whose author is a physicist who knew Einstein. Sensitive and stirring.

Whiteside, D.T., The Mathematical Papers of Isaac Newton, Volume VIII, 1697-1722. Cambridge University Press, 1981.

This last volume of Newton's papers contains commentary and documents on the brachistochrone problem, as well as: "a full report--on the rarely humorous, often secretive and vicious, ultimately sterile conflict over coveted priorities [on the Calculus] and imagined plagarisms which broke out publicly in 1712 and lasted, with much useless scuffling and furtive jockeying for position, for the next decade (and maybe is still not quite at an end)" (p. 30). See pp. 469-538 for Whiteside's commentary. (Thanks to Fred Rickey for this contribution.)

Ruckle, W.H., Geometric Games and Their Applications, Pitman, 1983; iii + 187 pp, $19.95 (P).

Entertaining treatment of those geometric games which offer partial information and can be given readily in normal form (thus excluding jigsaw puzzles, tic-tac-toe, chess, etc.). The mathematics used is only calculus and probability; assorted unsolved problems are included.

Stewart, B.M., Adventures Among the Toroids: A Study of Orientable Polyhedra with Regular Faces, Revised Second Edition, from the author (4494 Wausau Road, Okemos, MI 48864); v + 256 pp, $12.50 (P) postpaid.

A highly unusual book has re-emerged. Even with its new horizontal two-column format, it still won't fit in your bookcase; but you will treasure it for its imaginative subject matter, thousands of figures, and elegant hand-lettering in Chancery Script. This new edition has 50 pages of new material.

Schoenberg, Isaac J., Mathematical Time Exposures, MAA, 1982; ix + 270 pp, $30, $18 (P).

"In 1939 Hugo Steinhaus published his admirable Mathematical Snapshots... . My present aims are roughly similar, but the pace is more leisurely; my lens is not as fast as Steinhaus's." Schoenberg enlarges several snapshots of Steinhaus's, devotes four each of the 18 exposures to finite Fourier series and the motions of a billiard ball, and stays with geometry for most of the remainder. The spirit may be akin to Steinhaus's, but the style is that of the MAA Mathematical Gems series. Take this book on your next vacation.

Gardner, Martin, Wheels, Life and Other Mathematical Amusements, Freeman, 1983; ix + 261 pp, (P).

This is the 10th superb collection and update of Gardner's *Mathematical Games* columns from Scientific American, with these columns dating from 1970-72. Three of the 22 recreations deal with Conway's game of Life, with the third written especially for this volume. Most college students today have never heard of Gardner or seen his work. A few words of recommendation, plus a request to the local bookstore to stock a few titles, may stimulate the curiosity, enthusiasm, and delight of a new generation.

Gubbins, S., *et al.* (eds.), Statistics at Work: A Handbook of Statistical Studies for the Use of Teachers and Students, New Zealand Statistical Association (Box 1731, Wellington, N.Z.), 1982; 111 pp, (P).

Intended mainly as a resource book for teachers of senior-high courses in statistics, this book has 11 studies with real data, and suggestions for class exercises, projects, and discussion. Topics include lotteries, industrial sampling, earthquake risk, delinquency, mortality, election polls, agriculture, and others, embracing probability theory, social statistics, sample surveys, and design of experiments.

Larson, Loren C., Problem Solving Through Problems, Springer-Verlag, 1983; xi + 332 pp, $34 (P).

First-rate anthology of mathematical problem-solving techniques, destined to be the bible of problem-solvers, and a classic as influential as Pólya's How to Solve It. Over 700 problems, more than a third of them solved, have been taken from Olympiads, Putnam Exams, *The American Mathematical Monthly*, *Crux Mathematicorum*, *Mathematics Magazine*, and other sources. Most important, the problems are organized by technique and mathematical idea involved. No student or mathematician can fail to learn something beautiful from this book.

UMAP Modules 1982: Tools for Teaching, COMAP, 1983; ix + 544 pp, (P).

Another annual collection of self-contained, lesson-length instructional units on applications of mathematics and statistics. This year's trove features modules on whales, finding Laplace and Fourier transforms, review of cases by the U. S. Supreme Court, the budgetary process, card tricks, Simpson's paradox, and other topics; 23 modules in all.

Chatterji, S.D., *et al.* (eds.), Jahrbuch Überblicke Mathematik 1983, Birkhäuser, 1983; 223 pp, $19.95 (P).

Why review a book here whose title is obviously in German? For one thing, this book is the 16th volume of an excellent series of annals of mathematical surveys. For another, a quarter of the contents is in English. Finally, U. S. mathematical sciences majors may be reminded of the importance of language study by an occasional encouragement that valuable articles are written in other languages. This volume surveys Ski's 1980 counterexample to Hilbert's 16th Problem (on limit cycles of differential equations), new measures of information, Padé approximants, caustics and catastrophes, generalized matrix inverses, lattice point problems, and repeatability in probability. Short pieces cover accuracy of computer calculations, mathematicians and the French revolution, the work of Rado, and Euler as a teacher. But I won't tell you which articles are in English.

Zweng, Marilyn, *et al.* (eds.), Proceedings of the Fourth International Congress on Mathematical Education, Birkhäuser, 1983; xv + 725 pp, $70.

Contains edited versions of the almost 300 papers submitted to the Congress; altogether a rich and impressive compendium of research, advice, and speculation. (Unfortunately, misprints occur every paragraph or two; and most of the contributions in French and Spanish appear without benefit of diacritical marks.)

Pine, Eli S., How to Enjoy Calculus, with Computer Applications, Revised 3rd Ed., Steinlitz-Hammacher Co. (Box 187, Hasbrouck Heights, NJ 07604), 1983; ii + 155 pp, $7.95 (P).

This compact paperback has sold 200,000 copies since 1975, more than any single calculus textbook. Why? It offers a conversational and enthusiastic tone, helpful attitude, large print, small and uncrowded pages, and simple explanations. This new edition adds a couple of simple computer applications, plus a derivation of the derivative of a monomial.

Grossman, Michael, Bigeometric Calculus: A System with a Scale-Free Derivative, Archimedes Foundation (Box 240, Rockport, MA 01966), 1983; vii + 100 pp, (P).

Classical calculus is based on differences and offers an origin-free derivative. The bigeometric calculus presented here is based on ratios and offers a scale-free derivative. This book compares the two, shows their relationship, and suggests applications for which the latter may be more appropriate.

NEWS & LETTERS

LESTER R. FORD AWARDS

Three authors of articles published in the *American Mathematical Monthly* during 1982 were awarded Lester R. Ford prizes at the annual meeting of the MAA on January 27, 1984 in Louisville. The awards of $200 each recognize excellence in expository writing. The recipients were:

Robert F. Brown, "The Fixed Point Property and Cartesian Products", *American Mathematical Monthly*, 89 (1982), no. 9, pp. 654-678.

Tony Rothman, "Genius and Biographers: The Fictionalization of Evariste Galois", *American Mathematical Monthly*, 89 (1982) no. 2, pp. 84-106.

Robert S. Strichartz, "Radon Inversion-Variations on a Theme", *American Mathematical Monthly*, 89 (1982) no. 6, pp. 377-384.

SSS GEOMETRY AWARD SHARED

Three mathematicians shared a $2,500 award by Science Software Systems for their solutions to a geometry problem on the properties of the inverses of curves that self-invert at 90°. The winners are: J. F. Rigby, Univ. of Cardiff, *The Inversive Symmetry of Curves*, J. B. Wilker, Univ. of Toronto, *Möbius Equivalence and Euclidean Symmetry*, and W. Wunderlich, Vienna Technical Univ., *Congruent-Inverse Curve Pairs*.

GETTING PUBLISHED

Mathematicians want to "get published" and editors welcome publishable manuscripts. How does the potential author realize that potential? For some answers, read "How to Publish Mathematics", by R.P. Boas (former editor of the *Am. Math. Monthly*), in the Jan.-Feb. 1984 AWM Newsletter, and "Getting Published", in the Jan.-Feb. 1984 issue of *FOCUS*.

1983 W.L. PUTNAM COMPETITION

A-1. How many positive integers n are there such that n is an exact divisor of at least one of the numbers

$$10^{40},\ 20^{30}\ ?$$

A-2. The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

A-3. Let p be in the set $\{3, 5, 7, 11, \ldots\}$ of odd primes and let

$$F(n) = 1 + 2n + 3n^2 + \ldots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

A-4. Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero.

As usual, $\binom{m}{r} = \frac{m}{r!(m-r)!}$.

A-5. Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n.

Here $[x]$ denotes the greatest integer less than or equal to x.

A-6. Let $\exp(t)$ denote e^t and

$$F(x) = \frac{x^4}{\exp(x^3)} \int_0^x \int_0^{x-u} \exp(u^3+v^3)\, dv\, du.$$

Find $\lim_{x \to \infty} F(x)$ or prove that it does not exist.

B-1. Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v than to any other vertex of the cube. Find the volume of R.

B-2. For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

8, 4+4, 4+2+2, 4+2+1+1, and 2+2+2+1+1.

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n; here $[u]$ denotes the greatest integer less than or equal to u.

B-3. Assume that the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0.$$

has solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ on the whole real line such that

$$y_1^2(x) + y_2^2(x) + y_3^2(x) = 1$$

for all real x. Let

$$f(x) = (y_1'(x))^2 + (y_2'(x))^2 + (y_3'(x))^2.$$

Find constants A and B such that $f(x)$ is a solution to the differential equation

$$y' + A\,p(x)y = B\,r(x).$$

B-4. Let $f(n) = n + [\sqrt{n}]$ where $[x]$ is the largest integer less than or equal to x. Prove that, for every positive integer m, the sequence

$$m,\ f(m),\ f(f(m)),\ f(f(f(m))),\ \ldots$$

contains at least one square of an integer.

B-5. Let $||u||$ denote the distance from the real number u to the nearest integer. (For example, $||2.8|| = .2 = ||3.2||$.) For positive integers n, let

$$a_n = \frac{1}{n}\int_1^n \left|\left|\frac{n}{x}\right|\right| dx.$$

Determine $\lim_{x \to \infty} a_n$. You may assume the identity

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots = \frac{\pi}{2}.$$

B-6. Let k be a positive integer, let $m = 2k + 1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that

$$(P(r))^2 + (Q(r))^2 = -1.$$

SOLUTIONS TO 1983 INTERNATIONAL MATHEMATICAL OLYMPIAD

The following solutions to the 1983 IMO problems were prepared for publication in this Magazine *by Loren Larson and the St. Olaf College Problem Solving Group.*

1. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:

(i) $f(xf(y)) = yf(x)$ for all positive x, y;
(ii) $f(x) \to 0$ as $x \to +\infty$.

Sol. Let x be an arbitrary positive real number. By property (i),

$$f(xf(x)) = xf(x). \qquad (1)$$

It follows that

$$f(f(xf(x))) = f(xf(x)) = xf(x),$$

and setting $x = 1$,

$$f(f(f(1))) = f(1). \qquad (2)$$

On the other hand, setting $w = f(1)$,

$$f(f(w)) = f(1\cdot f(w)) = wf(1) = f(1)f(1),$$

and therefore

$$f(f(f(1))) = f(1)f(1). \qquad (3)$$

Comparing (2) and (3) we see that $f(1) = 1$.

Suppose that z is an arbitrary positive real number such that $f(z) = z$. Note that

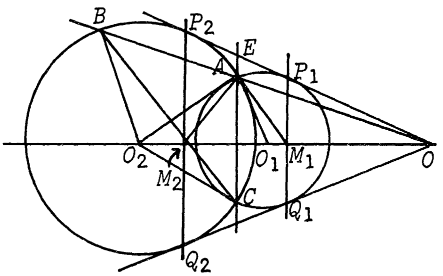
$$zf(1/z) = f((1/z)f(z)) = f(1) = 1,$$

so that $f(1/z) = 1/z$. An easy induction argument shows that

$f(z^n) = z^n$ for all integers n. If $z > 1$, then $f(z^n) \to \infty$ as $n \to \infty$; if $0 < z < 1$, $f(1/z^n) \to \infty$ as $n \to \infty$, and each of these contradict property (ii). Therefore the only number left fixed by f is 1. Thus, equation (1) implies that $xf(x) = 1$ for every x, or equivalently, the only function satisfying the given conditions is $f(x) \equiv 1/x$.

2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that the angles O_1AO_2 and M_1AM_2 are equal.

Sol. Let O be the intersection of P_1P_2 and Q_1Q_2; let OA intersect C_2 a second time at the point B. Let C denote the other point where C_1 and C_2 intersect, and let E denote the intersection of AC and P_1P_2.



By repeated use of the Pythagorean Theorem, or by knowing that tangents to two circles from any point on their radical axis are equal, one can show that $P_1E = P_2E$. It follows that line AC is the perpendicular bisector of M_1M_2, and therefore $AM_2 = AM_1$. This implies that $\angle O_1M_1A = \angle O_1M_2A$, or equivalently, that $\angle O_2M_2B = \angle O_1M_2A = \angle O_1M_2C$. The latter equality shows that B, M_2, C are collinear. Thus, $\Delta O_2CM \cong \Delta O_2AM_2$ and therefore $\angle O_2AM_2 = \angle O_2CM_2 = \angle O_2BM_2 = \angle O_1AM_1$. The result follows.

3. Let a, b, c be positive integers no two of which have a common divisor greater than 1. Show that

$$2abc - ab - bc - ca$$

is the largest integer which cannot be expressed in the form

$$xbc + yca + zab$$

where x, y, z are nonnegative integers.

Sol. Suppose that

$$2abc - ab - bc - ca = xbc + yca + zab$$

for some nonnegative integers x, y, z. Taken modulo a, this implies that $bc(x + 1) \equiv 0 \bmod a$. Since x is nonnegative and bc is relatively prime to a, it must be the case that $x \geq a-1$. In a similar manner, $y \geq b-1$ and $z \geq c-1$. Thus

$$\begin{aligned} &2abc - ab - bc - ca \\ &\geq (a-1)bc + (b-1)ca + (c-1)ab \\ &= 3abc - ab - bc - ca \end{aligned}$$

and this is a contradiction.

We claim that every integer r greater than $bc - b - c$ can be expressed in the form $yb + zc$, for some nonnegative integers y and z. To see this, note that since b and c are relatively prime, one of $0, c, 2c, \ldots, (b-1)c$ is congruent to r modulo b, say $r \equiv zc \bmod b$, $0 \leq z \leq b-1$. Note that $r - zc > bc - b - c - zc \geq bc - b - c - (b-1)c = -b$. This means that $r - zc \geq 0$, and thus there is a nonnegative integer x such that $r - cz = xb$; this establishes the claim.

Now suppose that

$$n > 2abc - ab - bc - ca.$$

Since a and bc are relatively prime, one of $0, bc, 2bc, \ldots, (a-1)bc$ is congruent to n modulo a, say

$$n \equiv xbc \bmod a, \ 0 \leq x \leq a-1.$$

Then $n - xbc = ra$ for some integer r. But $n - xbc > 2abc - ab - bc - ca - (a-1)bc = (bc - b - c)a$. From this, it follows that $r > bc - b - c$. But then, from the preceding paragraph, we know that there are nonnegative integers y and z such that $r = yb + zc$. With this substitution, $n - xbc = (yb + zc)a$ and the result follows.

4. Let ABC be an equilateral triangle, and E be the set of all points contained in the three segments AB, BC, and CA (including A, B, and C). Determine whether for every partition of E into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

Sol. Partition E into "red points" and "blue points". We may suppose that each side of the triangle contains at least two points of each color (otherwise it is easy to find a triangle of the required type).

Choose points D,E,F on AB, BC, CA respectively, so that

$$\frac{AD}{DB} = \frac{BE}{EC} = \frac{CF}{FA} = \frac{1}{2} .$$

Each of the sides of ΔDEF are perpendicular to one of the sides of ΔABC. Two of the vertices of ΔDEF have the same color -- say points S and T. Let XY denote the side of ΔABC that is perpendicular to ST. Then choose a point U on XY, different from either S or T, of the same color as S and T. Then ΔSTU is the required triangle.

5. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression?

Sol. Choose numbers as follows. Let $S_1 = \{1\}$, and $T_1 = S_1$. For $n > 1$, recursively define set S_n as

$$S_n = \{3^{n-2} + x : x \in T_{n-1}\}, \text{ and}$$

$$T_n = S_n \cup T_{n-1} .$$

An easy induction shows that T_n does not contain three numbers in arithmetic progression. Also, it is straightforward to show that T_n contains exactly 2^{n-1} elements, the largest of which is $(3^{n-1} + 1)/2$.

Thus the answer to the question is yes: T_{12} is an example of such a set.

6. Let a, b, and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

Determine when equality holds.

Sol. By symmetry, it suffices to consider two cases: $a \leq b \leq c$ and $c \leq b \leq a$.

Case 1. Suppose that $a \leq b \leq c$. Choose $x \geq 0$, $y \geq 0$ so that $b = a + x$ and $c = a + x + y$. Direct substitution into the above equation yields

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a)$$

$$= a^2xy + a^2y^2 + axy^2 + ay^3 + x^3(a-y)$$

$$+ x^2(a^2 - y^2).$$

But a, b, c are the sides of a triangle so that $a + b > c$, and this means that $a > y$. This completes the proof in this case. Also, we see that in this case we get equality iff $x = y = 0$, or iff the triangle is equilateral.

Case 2. Suppose that $c \leq b \leq a$. Choose $x > 0$, $y > 0$ so that $b = c + x$, $a = c + x + y$. Just as in Case 1, direct substitution establishes the result, with equality iff the triangle is equilateral.

CAPTIONS: THE COMPUTER-AGE GENERATION

Winning captions for the cartoon in this *Magazine*, January 1984, p. 40, will be published in our May issue. Correction: Vic Norton is at Bowling Green State University.

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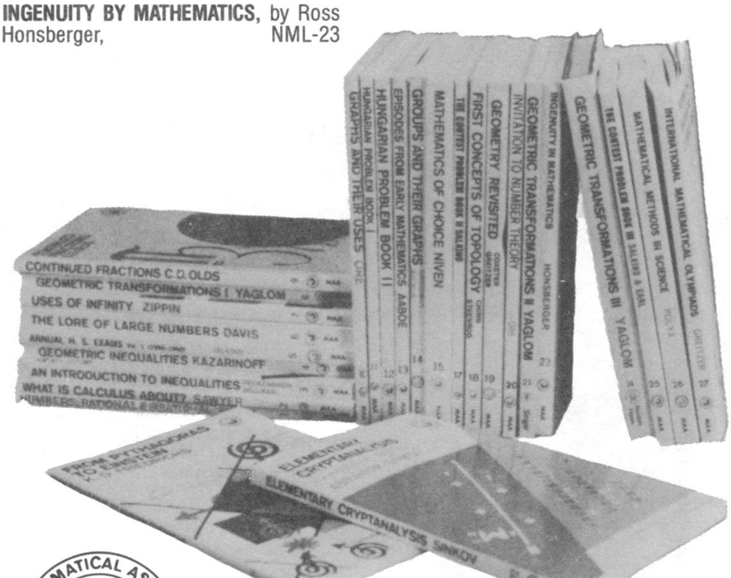

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MATHEMATICS MAGAZINE VOL. 57, NO. 2, MARCH 1984